

# Formal Verification of Conflict Detection Algorithms for Arbitrary Trajectories

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## Abstract

This paper presents an approach for developing formally verifiable conflict detection algorithms for aircraft flying arbitrary, nonlinear trajectories. The approach uses a multivariate polynomial global optimization algorithm based on Bernstein polynomials. Since any continuous function on a closed interval, such as an aircraft trajectory within a closed interval of time, can be uniformly approximated by a Bernstein polynomial, this global optimization algorithm can be used to define conflict detection algorithms for arbitrarily complicated trajectories. Conflict detection algorithms developed using this approach can be formally verified in a mechanical theorem prover. This represents an improvement over standard approaches to conflict detection for complex trajectories that essentially search for conflicts by testing many future states and are therefore not guaranteed to detect a given conflict. The proposed approach is illustrated with a formally verified conflict detection algorithm.

**Keywords:** Aircraft Conflict Detection, Formal Verification, Global Optimization  
**AMS subject classifications:** 65-00

# 1 Introduction

In air traffic management, a loss of separation is a violation of the separation requirement between two aircraft. This separation requirement is given by a minimum horizontal separation, e.g., 5 nautical miles, and a minimum vertical separation, e.g., 1000 feet [13]. Assuming an aircraft trajectory model, conflict detection algorithms predict whether or not two aircraft will lose separation within some lookahead time, which is typically 5 minutes. When a conflict is detected, conflict resolution algorithms compute resolution maneuvers for the aircraft that maintain the required aircraft separation. Conflict detection and resolution (CD&R) systems are part of computer-based systems that assist pilots and air traffic controllers to maintain safety in the airspace by keeping aircraft separated. These separation assurance systems are critical elements of air/ground distributed operational concepts for the next generation of air traffic management systems such as the US's Next Generation of Air Traffic Systems (NGATS) [23] and Europe's Single European Sky ATM Research (SESAR).<sup>1</sup>

The internal logic of CD&R algorithms relies on the reported current state information of the aircraft, typically 3D position and velocity vectors, and an aircraft trajectory model that propagates the current state information for a given lookahead time. Several state propagation methods for CD&R systems have been proposed [8]. For example, state-based conflict detection algorithms use a linear projection of the current state of the aircraft. This simple aircraft trajectory model corresponds to a point mass moving along a straight line at constant speed. More sophisticated state propagation methods assume nonlinear trajectories or probabilistic trajectory models.

This paper concerns *formal verification* of conflict detection algorithms that model aircraft trajectories as continuous functions over real numbers. In the context of this paper, formal verification refers to a computer-checked mathematical proof that a given algorithm satisfy some safety properties, where the algorithm and the properties are expressed in a formal specification language. In this sense, an algorithm is not a computer program, but a mathematical abstraction of a computer program. Formal verification, as defined in this paper, concerns the functional and logical correctness of algorithms. The formal methods area known as *software verification* deals with correctness of computer programs by looking into the semantic details of programming languages. Software verification is out of the scope of this paper. In particular, this paper assumes that the arithmetic used in the description of conflict detection algorithms has *real number* semantics, as opposed to *floating-point number* semantics.

A safety property of conflict detection algorithms known as *soundness* states

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<sup>1</sup><http://www.eurocontrol.int/content/sesar-and-research>.

that all potential conflicts, according to a given aircraft trajectory model, are detected. In this sense, an algorithm that always returns true is sound. Of course, such an algorithm is not particularly useful. Another safety property, known as *completeness*, states that the algorithm only detects potential conflicts. Hence, the algorithm that always return true is sound but not complete. The notions of soundness and completeness are related to the notions of missed alerts and false alerts, respectively. A conflict detection algorithm that misses alerts negatively affects the safety case for separation assurance systems, since it may indicate that certain maneuvers are safe when they are not. False alerts also have safety implications as they may diminish the confidence that pilots and air traffic controllers have on these systems.

Properties such as soundness and completeness have been formally verified for CD&R algorithms that assume linear trajectories using mechanical theorem provers [10, 12]. A conflict resolution algorithm for curved trajectories has been formally verified using hybrid-model checking techniques [18]. Other type of trajectories, such as piece-wise linear trajectories admit analytical methods [6, 7] and thus, formal verification of these algorithms is feasible. Most CD&R algorithms that handle complicated trajectories either iterate an analytical method at specified discrete steps [2, 16] or rely on numerical approximation methods [1, 17, 20]. Formal verification of these kinds of algorithms is very difficult. In general, soundness and completeness of iterative algorithms cannot be inferred from soundness and completeness of the analytical methods that they iterate since some conflicts may be missed or not correctly solved for input values outside the specified discrete steps. Furthermore, CD&R algorithms based on numerical methods are neither sound nor complete unless the computation errors are correctly accounted for.

This paper presents a numerical approximation approach for designing provably sound and complete conflict detection algorithms for arbitrary aircraft trajectories. In this approach, conflict detection for arbitrary aircraft trajectories is expressed as a global optimization problem and then analytically solved using *Bernstein polynomials* [9]. Methods for global optimization based on Bernstein polynomials allow for the computation of lower and upper bounds for arbitrary continuous functions, e.g., the separation distance between two aircraft, to any desired precision. These bounds are guaranteed to be correct. This is in contrast to some numerical methods for global optimization such as genetic algorithms [5] that do not guarantee the correctness of their results.

An important feature of the mathematical development presented in this paper is that it has been specified and formally verified in the Prototype Verification System (PVS) [14]. PVS is a proof assistant that consists of a specification language, based on classical higher-order logic, and a mechanical theorem prover for this logic. The PVS specification language enables the precise definition of mathemat-

ical objects such as *functions* and *relations*, and the precise statement of logical formulas such as *lemmas* and *theorems*. Proofs of logical formulas can be mechanically checked using the PVS theorem prover, which guarantees that every proof step is correct and that all possible cases of a proof are covered. All propositions, lemmas and theorems presented in this paper have been mechanically checked in PVS for logical correctness.

The rest of the paper is organized as follows. The mathematical notation used in this paper and its relation to the formal language of PVS is explained in §2. The conflict detection problem and how this problem can be expressed as a global optimization problem is discussed in §3. As part of this work, Bernstein polynomials have been formally specified and their main properties have been verified in PVS. This formal development of Bernstein polynomials is described in §4. A method for finding bounds of the maximum and minimum values of functions that are defined as the maximum of two polynomials is presented in §5. That method, which uses Bernstein polynomials, is the core function of a verified conflict detection algorithm for arbitrary trajectories proposed in §6. The last section concludes this paper.

## 2 Mathematical Notation and PVS

The use of a formal language, e.g., in this case the specification language of PVS, enforces rigorous definitions of mathematical objects, where all dependencies are clearly specified. This level of rigor guarantees a very high confidence on the correctness of the results presented in this paper. However, this also makes the notation heavy and difficult to read for the non-expert reader. To make this development more accessible to the casual reader, the work presented here uses a simpler mathematical notation that does not require the reader to be familiar with the syntax or semantics of the PVS language.

Despite the use of standard mathematical notation in many places, this paper heavily emphasizes the abstract data structures and specific signatures (types) of the functions used in the algorithms. This includes writing many properties as explicit statements involving function calls. The reason for this is that the intended audience includes not just mathematicians interested in global optimization but also researchers in air traffic management who may want to implement a formally verified conflict detection algorithm whose output can be trusted. Such readers will be interested in specific definitions that can be directly translated to code.

The PVS specification language is *strongly typed*, i.e., all declarations are explicitly typed. This feature guarantees that all PVS functions are total and well-defined. For instance, a mathematical formula that includes a division needs to

make explicit the fact that the divisor is different from zero, otherwise the expression would be undefined. In PVS, these conditions are handled by a type system, which is enforced by the PVS type-checker. Since PVS type annotations tend to be verbose, formulas in this paper are implicitly typed, i.e., it is assumed that the type domain of variables is inferred from the context where the formula appears.

PVS is based on higher-order logic, so it supports the definition of functions that return functions or that have functions as arguments. The notation  $\lambda x: e$  represents an anonymous function that takes as input  $x$  and returns  $e$ . As discussed above, PVS is a strongly typed language and, therefore, the type of  $x$  has to be explicitly declared in PVS. This paper assumes that the type of  $x$  is inferred from the context. A function that returns another function is called a *parametric function* and its arguments are called *parameters*. By convention, parameters are sub-indicated, e.g., a parametric function  $f$  that given  $t, x$ , and  $y$  returns a function of type  $\mathbb{R} \rightarrow \mathbb{R}$  is denoted  $f_{t,x,y}$ , where  $t, x, y$  are the parameters of  $f$ . Since  $f_{t,x,y}$  has the type  $\mathbb{R} \rightarrow \mathbb{R}$ , the function application  $f_{t,x,y}(z)$  has the type  $\mathbb{R}$  for any  $z \in \mathbb{R}$ .

PVS provides numerical types for natural and real numbers that correspond to the mathematical sets  $\mathbb{N}$  and  $\mathbb{R}$ . In PVS,  $1/3 + 1/3 + 1/3$  is exactly equal to 1 and  $\text{sqrt}(2)$  refers to the unique positive real number such that  $\text{sqrt}(2)*\text{sqrt}(2)$  is exactly equal to 2. The sets of integer and real numbers are unbounded and it is possible to prove properties about arbitrary large or small numbers.

The PVS specification language is a mathematical language rather than a programming language, where algorithms are defined as mathematical functions. Therefore, functions do not have side effects and variables have the mathematical meaning of arbitrary constant values instead of memory cells as in programming languages. By convention, names of functions that are intended to have a logical meaning, i.e., *predicates*, are written in *italics*. Functions that are intended to be used as algorithmic procedures are written in **typewriter** font.

The PVS specification language supports data structures such as records and tuples. Since records in this paper are mainly used to specify outputs of algorithms, they are also written in **typewriter** font. Record types are defined as  $T = \mathbf{a}_1:T_1 \times \dots \times \mathbf{a}_n:T_n$ , where  $T$  is the name of the record type,  $\mathbf{a}_1 \dots \mathbf{a}_n$  are the names of the fields, and  $T_i$ , for  $1 \leq i \leq n$ , is the type of the field  $\mathbf{a}_i$ . Field access will be denoted using the dot symbol. The operator **with** is used to overwrite records, e.g., if  $\mathbf{t}$  is a record of type  $T$ , the record  $\mathbf{t}' = \mathbf{t} \text{ with } [\mathbf{a}_1 \leftarrow e, \mathbf{a}_n \leftarrow f]$  refers to the record that is equal to  $\mathbf{t}$  in every field except in  $\mathbf{a}_1$  and  $\mathbf{a}_n$ , where it has the values  $e$  and  $f$ , respectively, i.e.,  $\mathbf{t}'.\mathbf{a}_1 = e$ ,  $\mathbf{t}'.\mathbf{a}_n = f$ , and for all  $1 < i < n$ ,  $\mathbf{t}'.\mathbf{a}_i = \mathbf{t}.\mathbf{a}_i$ .

Tuples will be typed in lowercase **boldface**, e.g.,  $\mathbf{a} = (a_0, \dots, a_{m-1})$  is an  $m$ -tuple where subindices from 0 to  $m-1$  are used to denote particular elements of  $\mathbf{a}$ . Given a positive natural number  $m$ , the order  $<$  between  $m$ -tuples is defined

by  $\mathbf{a} < \mathbf{b}$  if and only if  $a_j < b_j$  for all  $0 \leq j < m$ . Similarly, the order  $\leq$  between  $m$ -tuples is defined by  $\mathbf{a} \leq \mathbf{b}$  if and only if  $a_j \leq b_j$  for all  $0 \leq j < m$ . The  $m$ -tuples  $\mathbf{0}$  and  $\mathbf{1}$  represent the tuples whose components are all 0 and all 1, respectively. A *bounded box*  $[\mathbf{a}, \mathbf{b}]$ , where  $\mathbf{a} < \mathbf{b}$ , denotes the set of  $m$ -tuples greater than or equal to  $\mathbf{a}$  and less than or equal to  $\mathbf{b}$ . The box  $[\mathbf{0}, \mathbf{1}]$  is called the *unit box*.

As in the case of records, tuples can be overwritten using the operator **with**, e.g., if  $\mathbf{a}$  is an  $m$ -tuple, the tuple  $\mathbf{a}$  **with**  $[i \leftarrow b]$ , with  $0 \leq i < m$ , is equal to  $\mathbf{a}$  in every index except in  $i$  where it has the value  $b$ .

Algorithms for conflict detection considered in this paper use a 3D flat earth projection of the airspace. Aircraft positions and velocities in this 3D rectangular coordinate system are represented in PVS as 3-tuples in  $\mathbb{R}^3$ , where the first and second components denote the horizontal plane and the third component denotes the vertical dimension. Such tuples are appropriately called *vectors*. Components of vectors are sub-indicated by  $x$ ,  $y$ , and  $z$  instead of 0, 1, and 2. Furthermore, if  $\mathbf{w}$  is the vector  $(w_x, w_y, w_z)$ , then  $\mathbf{w}_{(x,y)}$  denotes the projection of  $\mathbf{w}$  in the horizontal plane, i.e.,  $\mathbf{w}_{(x,y)} = (w_x, w_y)$ , and  $\mathbf{w}^\perp$  denotes the vector  $(w_y, -w_x, w_z)$ . The notation  $\|\mathbf{w}\|$  refers to the norm of the vector  $\mathbf{w}$  and the notation  $\mathbf{w} \cdot \mathbf{w}'$  refers to the dot product of the vectors  $\mathbf{w}$  and  $\mathbf{w}'$ .

The work presented in this paper is part of a research project by the authors that aims at the development of verification technology for global optimization problems.<sup>2</sup> This research project includes the development of a PVS library for Bernstein polynomials, which is available as part of the PVS NASA Libraries.<sup>3</sup>

### 3 Conflict Detection

This section provides a mathematical description of aircraft conflicts for arbitrary trajectories and discusses how the problem of detecting aircraft conflicts can be expressed as a global optimization problem. Since conflicts between multiple aircraft can be detected in a pairwise fashion, only two aircraft are considered. These two aircraft are referred to as the *ownship* and the *intruder*.

The airspace volume is modeled using a flat-earth projection in a 3D rectangular system, i.e., aircraft positions are viewed as points in  $\mathbb{R}^3$ . In this airspace, the separation requirement between two aircraft is specified as a minimum horizontal separation  $D$  and a minimum vertical separation  $H$ . Typically,  $D$  is 5 nautical miles and  $H$  is 1000 feet [13]. In this paper,  $D$  and  $H$  are considered to be known numerical constants. The separation requirement can be understood as an imaginary horizontal cylinder, called *protected zone*, of height  $2H$  and radius  $D$  around

<sup>2</sup><http://shemesh.larc.nasa.gov/people/cam/Bernstein>.

<sup>3</sup><http://shemesh.larc.nasa.gov/fm/ftp/larc/PVS-library/pvslib.html>.

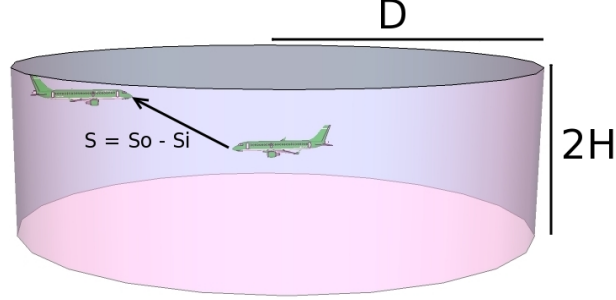


Figure 1: Loss of Separation

the intruder aircraft.

A loss of separation between the ownship and the intruder aircraft occurs when the horizontal distance between the aircraft is less than  $D$  and the vertical distance is less than  $H$ , i.e., when the ownship is in the interior of the intruder's protected zone. Let  $\mathbf{s}_o \in \mathbb{R}^3$  and  $\mathbf{s}_i \in \mathbb{R}^3$  be the current positions of the ownship and intruder aircraft, respectively. Formally, the ownship and intruder aircraft are said to be in *loss of separation* if the following predicate on  $D$ ,  $H$ ,  $\mathbf{s}_o$ , and  $\mathbf{s}_i$ , holds.

$$\text{loss}(D, H, \mathbf{s}_o, \mathbf{s}_i) \equiv |s_z| < H \text{ and } \|\mathbf{s}_{(x,y)}\| < D, \text{ where } \mathbf{s} = \mathbf{s}_o - \mathbf{s}_i.$$

Loss of separation is illustrated by Figure 1.

### 3.1 Trajectories

An aircraft trajectory represents the set of possible positions for the aircraft within a lookahead time  $T$  according to some state estimation model [8]. As in the case of  $D$  and  $H$ ,  $T$  is assumed to be a known numerical constant.

A state estimation model for CD&R systems may be as simple as a linear projection of the current position at the current constant speed. More complicated models consider uncertainties in the aircraft state due to aircraft dynamics, weather patterns, and other factors. In this paper, an *aircraft trajectory* is a continuous function of type  $\mathbb{R}^m \rightarrow \mathbb{R}^3$ , where the first variable in  $\mathbb{R}^m$  represents time and it is bounded by the time interval  $[0, T]$ . The other variables in  $\mathbb{R}^m$  represent uncertainties and are assumed to be bounded as well. Given  $\mathbf{x} \in \mathbb{R}^m$ , the output of a trajectory evaluated at  $\mathbf{x}$  is a point in  $\mathbb{R}^3$  that represents a 3-D position for the aircraft. The following examples give formal definitions of trajectories for several types of state estimation models.

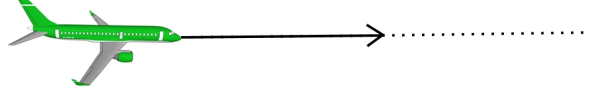


Figure 2: Linear Trajectory

**Example 1** (Linear Dynamics Without Uncertainty). *The simplest trajectory model for an aircraft is a linear projection of its current position  $\mathbf{s} \in \mathbb{R}^3$  along its current velocity  $\mathbf{v} \in \mathbb{R}^3$ . This type of trajectory can be represented by the parametric function  $\mathbf{linear}_{\mathbf{s},\mathbf{v}}: \mathbb{R} \rightarrow \mathbb{R}^3$ , with parameters  $\mathbf{s}$  and  $\mathbf{v}$ , defined by*

$$\mathbf{linear}_{\mathbf{s},\mathbf{v}}(t) \equiv \mathbf{s} + t \mathbf{v}. \quad (1)$$

*In this case,  $\mathbb{R}^m = \mathbb{R}$ , i.e.,  $m = 1$ , and the variable  $t \in [0, T]$  represents time. This linear trajectory is illustrated by Figure 2.*

**Example 2** (Linear Dynamics With Cross-Track Uncertainty). *In Example 1, the position at time  $t$  of an aircraft along a linear trajectory at constant speed is given by  $\mathbf{linear}_{\mathbf{s},\mathbf{v}}(t)$ , where  $\mathbf{s}$  and  $\mathbf{v}$  are the initial position and velocity vector of the aircraft. Uncertainty in the horizontal position of the aircraft is known as cross-track uncertainty [7]. If this uncertainty is bounded by some distance  $E_v$ , then the difference between the actual position and the position only considering cross-track uncertainty at any time is given by  $x \frac{E_v}{\|\mathbf{v}(x,y)\|} \mathbf{v}^\perp$ , for some  $x \in [-1, 1]$ . This difference vector is perpendicular to the initial velocity of the aircraft. Considering cross-track uncertainty, the trajectory of an aircraft along a linear path can be represented by the parametric function  $\mathbf{linearunc}_{\mathbf{s},\mathbf{v},E_v}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ , with parameters  $\mathbf{s}$ ,  $\mathbf{v}$ , and  $E_v$ , defined by*

$$\mathbf{linearunc}_{\mathbf{s},\mathbf{v},E_v}(t, x) \equiv \mathbf{s} + t \mathbf{v} + x \frac{E_v}{\|\mathbf{v}(x,y)\|} \mathbf{v}^\perp, \quad (2)$$

*where the variable  $t \in [0, T]$  represents time and the variable  $x \in [-1, 1]$  represents the (often unknown) cross-track uncertainty. This trajectory is illustrated by Figure 3.*

**Example 3** (Turn Dynamics Without Uncertainty). *During a steady coordinated turn without friction, the position of an aircraft will follow a circle of radius  $\frac{\nu^2}{g \tan \phi}$ , where  $\nu$  is the true air speed,  $g$  is the acceleration of gravity, and  $\phi$  is the bank angle of the aircraft. Thus, the trajectory of an aircraft during a turn can be represented*



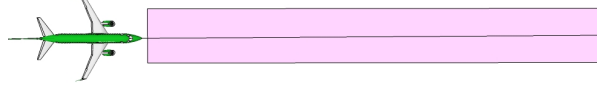


Figure 3: Linear Trajectory with Cross-Track Uncertainty

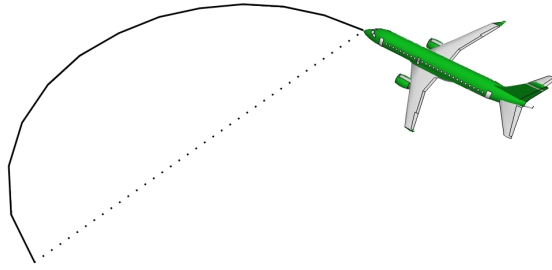


Figure 4: Turning Trajectory for Constant Bank-Angle

by the parametric function  $\text{turn}_{\mathbf{s},R,\alpha,\omega,v_z} : \mathbb{R} \rightarrow \mathbb{R}$ , with parameters  $\mathbf{s}$ ,  $R$ ,  $\alpha$ ,  $\omega$ , and  $v_z$ , defined by

$$\text{turn}_{\mathbf{s},R,\alpha,\omega,v_z}(t) \equiv \mathbf{s} + (R \sin(\alpha + t\omega), R \cos(\alpha + t\omega), t v_z), \quad (3)$$

where  $\mathbf{s}$  is the center point of the turn,  $\omega = \pm g \tan \frac{\phi}{v}$ ,  $\alpha$  is the angle along the turn at time zero,  $R = \frac{v^2}{g \tan \phi}$ , and  $v_z$  is the vertical speed. The variable  $t \in [0, T]$  represents time. This trajectory is illustrated by Figure 4.

**Example 4** (Turn Dynamics With Wind and Altitude Uncertainty). In Example 3, the position at time  $t$  of an aircraft in a steady turn is given by  $\text{turn}_{v,g,\phi,\mathbf{s},\alpha}(t)$ . In the event of a wind where its speed components are varying between 0 and  $E_w$  and assuming that the uncertainty in the altitude of the aircraft is bounded by  $E_a$ , the trajectory of an aircraft during a steady turn can be represented by the parametric function  $\text{windturn}_{\mathbf{s},R,\alpha,\omega,v_z,E_w,E_a} : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ , with parameters  $\mathbf{s}$ ,  $R$ ,  $\alpha$ ,  $\omega$ ,  $v_z$ ,  $E_w$ , and  $E_a$ , defined by

$$\begin{aligned} \text{windturn}_{\mathbf{s},R,\alpha,\omega,v_z,E_w,E_a}(t, x, y, z) \equiv & \text{turn}_{\mathbf{s},R,\alpha,\omega,v_z}(t) + \\ & (t x E_w, t y E_w, z E_a), \end{aligned} \quad (4)$$

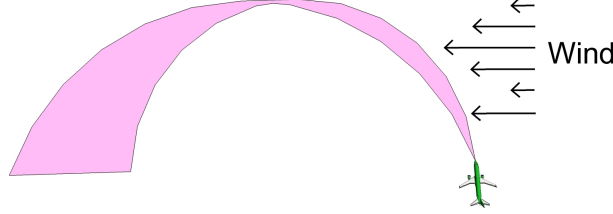


Figure 5: Turning Trajectory with Wind Uncertainty

where the variable  $t \in [0, T]$  represents time, the variables  $x, y \in [-1, 1]$  represent wind uncertainty, and the variable  $z \in [-1, 1]$  represents altitude uncertainty. A top-down view of this trajectory is illustrated by Figure 5.

As illustrated by these examples, the first variable in a trajectory function represents time and is bounded by  $[0, T]$ . If the trajectory function has more than one variable in the domain, the other variables are bounded in predetermined intervals as well. For instance, in Example 2 the variables  $x$  and  $y$ , representing along-track and cross-track errors, are bounded in the interval  $[-1, 1]$ .

Trajectories for the ownship and intruder aircraft are denoted by  $P_o$  and  $P_i$ , respectively. Without loss of generality, it can be assumed that these trajectories have the same variables, that is, they are both functions of type  $\mathbb{R}^m \rightarrow \mathbb{R}^3$ . Indeed, if one trajectory has arguments that are not used by the other trajectory, the other trajectory can be written as a function on its own arguments and those extra arguments as well. For instance, the trajectory  $\text{turn}_{\mathbf{s}, R, \alpha, \omega, v_z}$ , which has the type  $\mathbb{R} \rightarrow \mathbb{R}^3$ , can be trivially extended to a trajectory of type  $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ , the same type as  $\text{windturn}_{\mathbf{s}, R, \alpha, \omega, v_z, E_w, E_a}$ , by defining  $\text{turn}'_{\mathbf{s}, R, \alpha, \omega, v_z, E_w, E_a}(t, x, y, z) \equiv \text{turn}_{\mathbf{s}, R, \alpha, \omega, v_z}(t)$ .

### 3.2 Conflicts and Conflict Detection Algorithms

While loss of separation is formalized as a predicate on two aircraft positions  $\mathbf{s}_o$  and  $\mathbf{s}_i$ , a conflict between two aircraft is formalized as a predicate on the ownship and intruder trajectories,  $P_o$  and  $P_i$  in  $\mathbb{R}^m \rightarrow \mathbb{R}^3$ , respectively, and a box  $[\mathbf{a}, \mathbf{b}] \in \mathbb{R}^m$  that represents a range of interest in  $\mathbb{R}^m$ , e.g., time should be in  $[0, T]$ . The trajectories  $P_o$  and  $P_i$  are in *conflict* on the box  $[\mathbf{a}, \mathbf{b}]$  if there exists a point  $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$  such that the positions  $P_o(\mathbf{x})$  and  $P_i(\mathbf{x})$  are in loss of separation:

$$\text{conflict?}(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i) \equiv \exists \mathbf{x} \in [\mathbf{a}, \mathbf{b}] : \text{los?}(D, H, P_o(\mathbf{x}), P_i(\mathbf{x})). \quad (5)$$

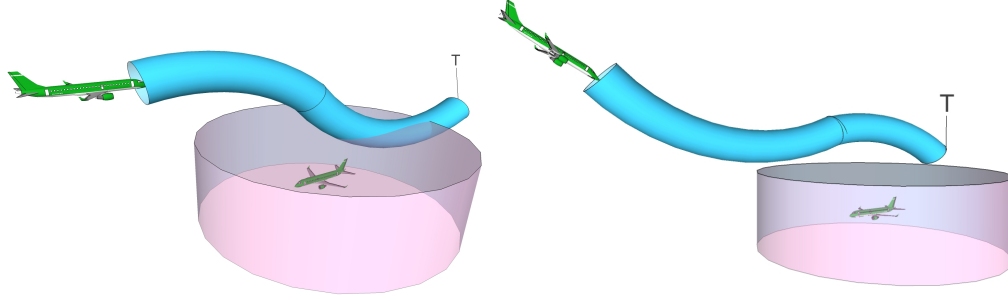


Figure 6: Conflict and Conflict-Free Scenarios

Figure 6 illustrates conflict and conflict-free scenarios in a relative coordinate system, where the intruder is at the origin of the system and the ownship is moving relative to the intruder. In each scenario, the “tube” in front of the ownship represents the relative trajectory of the aircraft for a given box  $[\mathbf{a}, \mathbf{b}] \in \mathbb{R}^m$ , where the first component in the box the time interval  $[0, T]$ . When  $\mathbb{R}^m = \mathbb{R}$ , i.e.,  $m = 1$ , the relative trajectory is a just a line.

An algorithm used by an aircraft to detect conflicts with another aircraft is called a *conflict detection algorithm*. In this paper, a conflict detection algorithm is a function  $\text{cd}$  that takes as inputs  $D, H, \mathbf{a}, \mathbf{b}, P_o$ , and  $P_i$  and returns an element of type  $\text{CDOutcome}$ , with possible values  $\text{LossAt}(\mathbf{c})$ , where  $\mathbf{c} \in [\mathbf{a}, \mathbf{b}]$ ,  $\text{NoConflict}$ , and  $\text{Unknown}$ .

Formally, a conflict detection algorithm  $\text{cd}$  is *sound* if for all positive real numbers  $D, H$ , boxes  $[\mathbf{a}, \mathbf{b}]$ , and trajectories  $P_o, P_i$ ,  $\text{cd}(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i) = \text{NoConflict}$  implies that  $\neg \text{conflict?}(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i)$  hold; it is *complete* if  $\text{cd}(\mathbf{a}, \mathbf{b}, P_o, P_i) = \text{LossAt}(\mathbf{c})$  for some  $\mathbf{c} \in [\mathbf{a}, \mathbf{b}]$  implies that  $\text{conflict?}(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i)$  holds. An algorithm that is *sound* and *complete* is said to be *correct*. It is clear from the definition that correctness is a key safety property for a conflict detection algorithm.

### 3.3 Conflict Detection and Global Optimization

Conflict between two trajectories can be written as an optimization problem and hence global optimization methods can be used for conflict detection. This problem transformation is accomplished through the cylindrical norm, which is described in detail in [11]. The *cylindrical norm* of a vector  $\mathbf{w}$  is the quantity

$$\|\mathbf{w}\|_{D,H} \equiv \max \left( \frac{|w_z|}{H}, \frac{\|\mathbf{w}_{(x,y)}\|}{D} \right). \quad (6)$$

With this length,  $\mathbb{R}^3$  is a metric space in the sense of real analysis [21], and it therefore satisfies the triangle inequality. The following lemma and theorems follow directly from definitions.

**Lemma 1.** *The ownship and the intruder, which have position vectors  $\mathbf{s}_o$  and  $\mathbf{s}_i$ , are in loss of separation, i.e.  $\text{los?}(D, H, \mathbf{s}_o, \mathbf{s}_i)$  holds, if and only if  $\|\mathbf{s}_o - \mathbf{s}_i\|_{D,H} < 1$ .*

Instead of the cylindrical norm, which involves the square root and absolute value functions, the square of the cylindrical norm is considered. The function  $\|\mathbf{w}\|_{D,H}^2$ , defined as follows

$$\|\mathbf{w}\|_{D,H}^2 \equiv \max \left( \frac{w_z^2}{H^2}, \frac{w_x^2 + w_y^2}{D^2} \right),$$

is a maximum of two polynomials, the first in  $w_z$ , and the second in  $w_x$  and  $w_y$ . The following theorem shows that conflict between two trajectories can be expressed using this function.

**Theorem 1.** *For all boxes  $[\mathbf{a}, \mathbf{b}]$ , positive constant values  $D$  and  $H$ , and trajectories  $P_o$  and  $P_i$ ,  $\text{conflict?}(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i)$  holds if and only if there exists  $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$  such that  $\text{sqdist}_{D,H,P_o,P_i}(\mathbf{x}) < 1$ , where*

$$\text{sqdist}_{D,H,P_o,P_i}(\mathbf{x}) \equiv \max \left( \frac{w_z^2}{H^2}, \frac{w_x^2 + w_y^2}{D^2} \right), \text{ with } \mathbf{w} = P_o(\mathbf{x}) - P_i(\mathbf{x}). \quad (7)$$

When both  $P_o$  and  $P_i$  are polynomial trajectories, as in the cases of Example 1 and Example 2, the parametric function  $\text{sqdist}$  is defined as the maximum between two polynomials. If the trajectories are not polynomials, such as the trajectories in Example 3 and Example 4, since these functions are continuous on  $[\mathbf{a}, \mathbf{b}]$  they can be uniformly approximated by polynomials to any given precision. This result is known as the Weierstrass Approximation Theorem [9].

The predicate  $\text{approx?}$  specifies whether a given trajectory  $P$  is approximated by a trajectory  $P'$  on the box  $[\mathbf{a}, \mathbf{b}]$  by a horizontal precision  $\varepsilon_D \geq 0$  and a vertical precision  $\varepsilon_H \geq 0$ .

$$\begin{aligned} \text{approx?}(\mathbf{a}, \mathbf{b}, \varepsilon_D, \varepsilon_H, P, P') &\equiv \forall \mathbf{x} \in [\mathbf{a}, \mathbf{b}]: |P(\mathbf{x})_z - P'(\mathbf{x})_z| < \varepsilon_H \text{ and} \\ &\quad \|P(\mathbf{x})_{(x,y)} - P'(\mathbf{x})_{(x,y)}\| < \varepsilon_D. \end{aligned}$$

If it is known that the trajectories  $P'_o$  and  $P'_i$ , which are approximations of  $P_o$  and  $P_i$ , respectively, are in conflict, then it is possible to determine if  $P_o$  and  $P_i$  are in conflict as well. This determination can be done by considering the effect that the differences between these trajectories and their approximations have on the output of the function  $\text{sqdist}$ . This effect is captured in the following proposition, which can be proved by basic algebraic manipulations.

**Proposition 1.** Let  $[\mathbf{a}, \mathbf{b}]$ ,  $P_o$ ,  $P'_o$ ,  $P_i$ ,  $P'_i$ ,  $\varepsilon_D = \varepsilon_{oD} + \varepsilon_{iD}$ , and  $\varepsilon_H = \varepsilon_{oH} + \varepsilon_{iH}$ , be such that  $\text{approx}^?(\mathbf{a}, \mathbf{b}, \varepsilon_{oD}, \varepsilon_{oH}, P_o, P'_o)$  and  $\text{approx}^?(\mathbf{a}, \mathbf{b}, \varepsilon_{iD}, \varepsilon_{iH}, P_i, P'_i)$  hold. Then, for all  $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ ,

1.  $\text{sqdist}_{D,H,P'_o,P'_i}(\mathbf{x}) < 1 - \delta_-(D, H, \varepsilon_D, \varepsilon_H)$  implies  $\text{sqdist}_{D,H,P_o,P_i}(\mathbf{x}) < 1$ ,
2.  $\text{sqdist}_{D,H,P'_o,P'_i}(\mathbf{x}) \geq 1 + \delta_+(D, H, \varepsilon_D, \varepsilon_H)$  implies  $\text{sqdist}_{D,H,P_o,P_i}(\mathbf{x}) \geq 1$ ,

where

$$\begin{aligned} \delta_-(D, H, \varepsilon_D, \varepsilon_H) &\equiv \max\left(2\frac{\varepsilon_D}{D} - \frac{\varepsilon_D^2}{D^2}, 2\frac{\varepsilon_H}{H} - \frac{\varepsilon_H^2}{H^2}\right), \\ \delta_+(D, H, \varepsilon_D, \varepsilon_H) &\equiv \max\left(2\frac{\varepsilon_D}{D} + \frac{\varepsilon_D^2}{D^2}, 2\frac{\varepsilon_H}{H} + \frac{\varepsilon_H^2}{H^2}\right). \end{aligned} \quad (8)$$

The next result follows directly from Theorem 1 and Proposition 1. It shows that if  $P_o$  and  $P_i$  are trajectories that are approximated by  $P'_o$  and  $P'_i$ , then range information for the function  $\text{sqdist}_{D,H,P'_o,P'_i}$  can be used to detect conflict between  $P_o$  and  $P_i$ . This result will be directly used later to detect conflicts between  $P_o$  and  $P_i$ , when  $P'_o$  and  $P'_i$  are polynomial trajectories, by approximating the range of  $\text{sqdist}_{D,H,P'_o,P'_i}$ . Since any trajectory can be uniformly approximated by polynomial trajectories, conflict detection for arbitrary trajectories can be reduced to computing range information for the function  $\text{sqdist}_{D,H,P'_o,P'_i}$  for some polynomial trajectories  $P'_o$  and  $P'_i$ .

**Proposition 2.** If  $\text{approx}^?(\mathbf{a}, \mathbf{b}, \varepsilon_{oD}, \varepsilon_{oH}, P_o, P'_o)$  and  $\text{approx}^?(\mathbf{a}, \mathbf{b}, \varepsilon_{iD}, \varepsilon_{iH}, P_i, P'_i)$ , then

1. If  $\mathbf{c}$  is a point in the box  $[\mathbf{a}, \mathbf{b}]$  and  $\text{sqdist}_{D,H,P'_o,P'_i}(\mathbf{c}) < 1 - \delta_-(D, H, \varepsilon_D, \varepsilon_H)$ , then  $\text{conflict}^?(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i)$ .
2.  $\text{conflict}^?(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i)$  implies  $\text{sqdist}_{D,H,P'_o,P'_i}(\mathbf{x}) \leq 1 + \delta_+(D, H, \varepsilon_D, \varepsilon_H)$  for some  $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ ,

Numerical approximation methods based on Bernstein polynomials are well-known by the global optimization community [3]. The next section describes a formalization of Bernstein polynomials in PVS, from an algorithmic point of view. A formally verified algorithm for finding range information for the maximum and minimum values of a function defined as the maximum of two polynomials is presented in §5. That algorithm is used in verified algorithms for conflict detection of arbitrary trajectories, proposed in §6, the correctness of which follows from Proposition 2.

## 4 Formalization of Bernstein Polynomials

Bernstein polynomials are mathematical objects used to approximate continuous functions. This section presents an algorithmic formalization of multivariate Bernstein polynomials and their main properties. All properties presented in this section have been formally proved in the PVS theorem prover for both univariate and multivariate polynomials.

Let  $\mathbf{i}$  be an  $m$ -tuple of natural numbers and  $\mathbf{x}$  be an  $m$ -tuple of variables over  $\mathbb{R}$ . The product  $\mathbf{x}^{\mathbf{i}} = x_0^{i_0} \cdots x_{m-1}^{i_{m-1}}$  is called an  $m$ -variable *monomial*. The  $m$ -tuple  $\mathbf{i}$  is called the *index* of the monomial  $\mathbf{x}^{\mathbf{i}}$ . An  $m$ -variable *polynomial* of degree  $\mathbf{n}$  is a finite sum of the form

$$p(\mathbf{x}) = \sum_{\mathbf{i} \leq \mathbf{n}} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}},$$

where the elements  $c_{\mathbf{i}} \in \mathbb{R}$  are called the *coefficients* of  $p$ . In PVS, an  $m$ -polynomial  $p$  is represented by a function from  $\mathbb{R}^m$  into  $\mathbb{R}$ .

Let  $p(\mathbf{x}) = \sum_{\mathbf{i} \leq \mathbf{n}} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$  be an  $m$ -variable polynomial. For any bounded box  $[\mathbf{a}, \mathbf{b}]$ , the  $m$ -polynomial  $p_{[\mathbf{a}, \mathbf{b}]}$  can be defined as follows

$$p_{[\mathbf{a}, \mathbf{b}]}(\mathbf{x}) \equiv \sum_{\mathbf{k} \leq \mathbf{n}} \sum_{\mathbf{k} \leq \mathbf{i} \leq \mathbf{n}} (c_{\mathbf{i}} \prod_{j=0}^{m-1} \binom{i_j}{k_j} (b_j - a_j)^{k_j} a_j^{i_j - k_j}) \mathbf{x}^{\mathbf{k}}. \quad (9)$$

The following proposition states that any multivariate polynomial on an arbitrary box can be transformed into a polynomial on the unit box.

**Proposition 3.** *Let  $[\mathbf{a}, \mathbf{b}]$  be a bounded box and  $p$  an  $m$ -polynomial, For all  $\mathbf{x} \in [0, 1]$ ,  $p(\sigma_{[\mathbf{a}, \mathbf{b}]}(\mathbf{x})) = p_{[\mathbf{a}, \mathbf{b}]}(\mathbf{x})$ , where*

$$\sigma_{[\mathbf{a}, \mathbf{b}]}(\mathbf{x})_j \equiv a_j + x_j(b_j - a_j). \quad (10)$$

Furthermore, since  $\sigma_{[\mathbf{a}, \mathbf{b}]}$  is a bijection, for all  $\mathbf{y} \in [\mathbf{a}, \mathbf{b}]$ ,  $p(\mathbf{y}) = p_{[\mathbf{a}, \mathbf{b}]}(\sigma_{[\mathbf{a}, \mathbf{b}]}^{-1}(\mathbf{y}))$ . The inverse  $\sigma_{[\mathbf{a}, \mathbf{b}]}^{-1}$  is given on the  $j$ -th component by  $\sigma_{[\mathbf{a}, \mathbf{b}]}^{-1}(\mathbf{y})_j = \frac{y_j - a_j}{b_j - a_j}$ .

The PVS formalization provides a function `translate` that takes as inputs the arrays  $\mathbf{a}$  and  $\mathbf{b}$ , and an  $m$ -polynomial  $p$ , and returns the polynomial  $p_{[\mathbf{a}, \mathbf{b}]}$  defined by Formula (9). Proposition 3 implies that the equalities

$$\begin{aligned} p(\sigma_{[\mathbf{a}, \mathbf{b}]}(\mathbf{x})) &= \text{translate}(\mathbf{a}, \mathbf{b}, p)(\mathbf{x}), \\ p(\mathbf{y}) &= \text{translate}(\mathbf{a}, \mathbf{b}, p)(\sigma_{[\mathbf{a}, \mathbf{b}]}^{-1}(\mathbf{y})), \end{aligned}$$

hold for all  $\mathbf{x}$  and  $\mathbf{y}$ . It follows directly from this that the maximum (resp. minimum) value attained by  $p$  on the box  $[\mathbf{a}, \mathbf{b}]$  is the maximum (resp. minimum) value attained by the polynomial `translate`( $\mathbf{a}, \mathbf{b}, p$ ) on the unit box.

## 4.1 Bernstein Basis

An  $m$ -variable Bernstein polynomial is an  $m$ -variable polynomial of the form

$$p(\mathbf{x}) = \sum_{\mathbf{i} \leq \mathbf{n}} \hat{b}_{\mathbf{i}} B_{\mathbf{n},\mathbf{i}}(\mathbf{x}), \quad (11)$$

where  $\hat{b}_{\mathbf{i}} \in \mathbb{R}$  and

$$B_{\mathbf{n},\mathbf{i}}(\mathbf{x}) \equiv \prod_{j=0}^{m-1} \binom{n_j}{i_j} x_j^{i_j} (1 - x_j)^{n_j - i_j}. \quad (12)$$

The coefficients  $\hat{b}_{\mathbf{i}}$  are called the Bernstein coefficients of  $p$ .

Proposition 4 states that any polynomial can be written as a polynomial in Bernstein form by a simple transformation. Thus, the  $m$ -variable polynomials  $B_{\mathbf{n},\mathbf{i}}(\mathbf{x})$  in Formula (12) form a basis for the vector space of  $m$ -variable polynomials of degree at least  $\mathbf{n}$ .

**Proposition 4.** *Any  $m$ -variable polynomial  $p(\mathbf{x}) = \sum_{\mathbf{i} \leq \mathbf{n}} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$  can be written in Bernstein form as  $p(\mathbf{x}) = \sum_{\mathbf{k} \leq \mathbf{n}} \hat{b}_{\mathbf{k}} B_{\mathbf{n},\mathbf{k}}(\mathbf{x})$ , where*

$$\hat{b}_{\mathbf{k}} \equiv \sum_{\mathbf{i} \leq \mathbf{k}} \left( c_{\mathbf{i}} \prod_{j=0}^{m-1} \frac{\binom{k_j}{i_j}}{\binom{n_j}{i_j}} \right).$$

In PVS,  $m$ -variable polynomials in Bernstein form are represented using the data structure proposed by Smith in [22]. The technical details of that representation are outside the scope of this paper. Since the method presented in this paper does not depend on a particular representation of polynomials, it suffices to say that the PVS formalization provides a function `tomultibern` that takes an  $m$ -variable polynomial  $p$  as input and returns an element of a particular data structure that represents the Bernstein form of  $p$  given by Proposition 4. Furthermore, a function `eval` that takes as inputs a representation of an  $m$ -variable polynomial in Bernstein form and an  $m$ -tuple  $\mathbf{x}$ , and returns a real number is defined such that for all  $\mathbf{x} \in \mathbb{R}^m$ ,

$$\text{eval}(\text{tomultibern}(p), \mathbf{x}) = p(\mathbf{x}). \quad (13)$$

## 4.2 Properties of Bernstein Polynomials

A key result that makes Bernstein polynomials useful for proving polynomial inequalities is that the Bernstein coefficients of a polynomial provide lower and upper bounds for the values of the polynomial over the unit box.

**Proposition 5.** Let  $p(\mathbf{x}) = \sum_{\mathbf{i} \leq \mathbf{n}} \hat{b}_{\mathbf{i}} B_{\mathbf{n},\mathbf{i}}(\mathbf{x})$  be an  $m$ -variable polynomial in Bernstein form,  $r$  be a real number, and  $\Re$  be a real order in  $\{\leq, <, \geq, >\}$ . If  $\hat{b}_{\mathbf{i}} \Re r$ , for all  $\mathbf{i} \leq \mathbf{n}$ , then  $p(\mathbf{x}) \Re r$ , for all  $\mathbf{x} \in [0, 1]$ .

Another useful property of Bernstein polynomials is that the values of the function at the endpoints of the unit box are Bernstein coefficients of the polynomial. Let  $\mathbf{n}$  be an  $m$ -tuple of natural numbers. The set  $\mathcal{C}_{\mathbf{n}}$  of  $m$ -tuples of natural numbers denotes the *endpoint indices* of  $\mathbf{n}$  and it is defined as follows.

$$\mathcal{C}_{\mathbf{n}} \equiv \{\mathbf{i} \leq \mathbf{n} \mid \forall 0 \leq j < m : i_j = 0 \text{ or } i_j = n_j\}. \quad (14)$$

**Proposition 6.** Let  $\bar{\mathbf{x}}$  be an  $m$ -tuple such that  $\bar{x}_j = 0$  or  $\bar{x}_j = 1$  for all  $0 \leq j < m$ . Given an  $m$ -tuple  $\mathbf{n}$  of natural numbers, define  $\bar{\mathbf{i}} \in \mathcal{C}_{\mathbf{n}}$  by  $\bar{i}_j = 0$  if  $\bar{x}_j = 0$  and  $\bar{i}_j = n_j$  if  $\bar{x}_j = 1$ . For any  $m$ -variable polynomial in Bernstein form  $p(\mathbf{x}) = \sum_{\mathbf{i} \leq \mathbf{n}} \hat{b}_{\mathbf{i}} B_{\mathbf{n},\mathbf{i}}(\mathbf{x})$ ,  $p(\bar{\mathbf{x}}) = \hat{b}_{\bar{\mathbf{i}}}$ .

By Proposition 5 and Proposition 6, the following inequalities hold.

$$\begin{aligned} \min_{\mathbf{i} \leq \mathbf{n}} \hat{b}_{\mathbf{i}} &\leq \min_{\mathbf{x} \in [0,1]} p(\mathbf{x}) \leq \min_{\mathbf{i} \in \mathcal{C}_{\mathbf{n}}} \hat{b}_{\mathbf{i}}, \\ \max_{\mathbf{i} \in \mathcal{C}_{\mathbf{n}}} \hat{b}_{\mathbf{i}} &\leq \max_{\mathbf{x} \in [0,1]} p(\mathbf{x}) \leq \max_{\mathbf{i} \leq \mathbf{n}} \hat{b}_{\mathbf{i}}. \end{aligned} \quad (15)$$

The record type

$$\begin{aligned} \text{Outminmax} &\equiv \text{lbmin}:\text{real} \times \text{lbmax}:\text{real} \times \text{lbvar}:\mathbb{R}^{m+} \times \\ &\quad \text{ubmax}:\text{real} \times \text{ubmin}:\text{real} \times \text{ubvar}:\mathbb{R}^{m+}, \end{aligned}$$

defined in PVS, stores information about the range of a function  $f: \mathbb{R}^m \rightarrow \mathbb{R}$  over a given box. The type  $\mathbb{R}^{m+}$  is the type of points in  $\mathbb{R}^m$  extended with special value that represents an empty tuple. The intended semantics of the type **Outminmax** is given by the predicate *sound?*. This predicate is defined on a function  $f: \mathbb{R}^m \rightarrow \mathbb{R}$ , an element **omm** of type **Outminmax**, and a box  $[\mathbf{a}, \mathbf{b}]$  such that *sound?*( $f, \text{omm}, \mathbf{a}, \mathbf{b}$ ) holds if and only if

- $\min_{\mathbf{x} \in [\mathbf{a}, \mathbf{b}]} f(\mathbf{x}) \in [\text{omm.lbmin}, \text{omm.lbmax}]$ ,  $f(\text{omm.lbvar}) = \text{omm.lbmax}$ , and
- $\max_{\mathbf{x} \in [\mathbf{a}, \mathbf{b}]} f(\mathbf{x}) \in [\text{omm.ubmin}, \text{omm.ubmax}]$ ,  $f(\text{omm.ubvar}) = \text{omm.ubmin}$ .

Thus, if *sound?*( $f, \text{omm}, \mathbf{a}, \mathbf{b}$ ) holds, then **omm** provides bounds on the minimum and maximum values of  $f$  on  $[\mathbf{a}, \mathbf{b}]$ .

The PVS formalization provides a function **berncoeffsminmax** that takes as input a representation of an  $m$ -variable polynomial in Bernstein form  $p(\mathbf{x}) = \sum_{\mathbf{i} \leq \mathbf{n}} \hat{b}_{\mathbf{i}} B_{\mathbf{n},\mathbf{i}}(\mathbf{x})$  and returns an element **omm** of type **Outminmax** such that



- $\text{omm.lbmin} = \min_{\mathbf{i} \leq \mathbf{n}} \hat{b}_{\mathbf{i}}$ ,  $\text{omm.lbmax} = \min_{\mathbf{i} \in \mathcal{C}_{\mathbf{n}}} \hat{b}_{\mathbf{i}}$ ,  $p(\text{omm.lbvar}) = \text{lbmax}$ , and
- $\text{omm.ubmin} = \max_{\mathbf{i} \in \mathcal{C}_{\mathbf{n}}} \hat{b}_{\mathbf{i}}$ ,  $\text{omm.ubmax} = \max_{\mathbf{i} \leq \mathbf{n}} \hat{b}_{\mathbf{i}}$ ,  $p(\text{omm.ubvar}) = \text{ubmin}$ .

The next proposition follows directly from the definition of `berncoeffsminmax`, Proposition 6, Proposition 4, and Formula (15).

**Proposition 7.** *For any  $m$ -variable polynomial  $p$ ,*

$$\text{sound?}(p, \text{berncoeffsminmax}(\text{tomultibern}(p)), \mathbf{0}, \mathbf{1})$$

*holds.*

Proposition 3 implies that the range of an  $m$ -variable polynomial  $p$  on the box  $[\mathbf{a}, \mathbf{b}]$  corresponds to the range of the  $m$ -variable polynomial in Bernstein form represented by `tomultibern(translate( $\mathbf{a}, \mathbf{b}, p$ ))` on the unit box  $[\mathbf{0}, \mathbf{1}]$ . When `berncoeffsminmax` is evaluated on this representation, the fields `lbvar` and `ubvar` of the resulting element of type `Outminmax` are points in  $[\mathbf{0}, \mathbf{1}]$  where the  $m$ -variable polynomial  $p$  attains the values `lbmax` and `ubmin`, respectively. Proposition 4 and Formula (13) imply that for any  $m$ -tuple  $\mathbf{x}$  in the unit box,

$$\text{eval}(\text{tomultibern}(\text{translate}(\mathbf{a}, \mathbf{b}, p), \mathbf{x}) = \text{translate}(\mathbf{a}, \mathbf{b}, p)(\mathbf{x}).$$

Therefore, Formula (10) and Formula (13) imply that the equations

$$\begin{aligned} \text{eval}(\text{tomultibern}(\text{translate}(\mathbf{a}, \mathbf{b}, p), \text{lbvar}) &= p(\sigma_{[\mathbf{a}, \mathbf{b}]}(\text{lbvar})), \\ \text{eval}(\text{tomultibern}(\text{translate}(\mathbf{a}, \mathbf{b}, p), \text{ubvar}) &= p(\sigma_{[\mathbf{a}, \mathbf{b}]}(\text{ubvar})) \end{aligned}$$

both hold.

**Proposition 8.** *For any  $m$ -variable polynomial  $p$  and box  $[\mathbf{a}, \mathbf{b}]$ , if*

$$\text{omm} = \text{berncoeffsminmax}(\text{tomultibern}(\text{translate}(\mathbf{a}, \mathbf{b}, p))),$$

*then  $\text{sound?}(p, \text{denorm\_omm}(\text{omm}, \mathbf{a}, \mathbf{b}), \mathbf{a}, \mathbf{b})$  holds, where*

$$\begin{aligned} \text{denorm\_omm}(\text{omm}, \mathbf{a}, \mathbf{b}) \equiv \text{omm with } [ \text{lbvar} \leftarrow \sigma_{[\mathbf{a}, \mathbf{b}]}(\text{omm.lbvar}), \\ \text{ubvar} \leftarrow \sigma_{[\mathbf{a}, \mathbf{b}]}(\text{omm.ubvar}) ]. \end{aligned}$$

According to Proposition 8, the range information for a polynomial  $p$  over an arbitrary box  $[\mathbf{a}, \mathbf{b}]$  can be computed using the functions `translate`, `tomultibern`, `berncoeffsminmax`, and `denorm_omm`.

### 4.3 Domain Subdivision

The reciprocal implication of Proposition 5 does not hold in general, i.e., the fact that a polynomial inequality holds on the unit box does not imply that the Bernstein coefficients of the polynomial satisfy the same inequality. In particular, the lower and upper bounds of the minimum and maximum values of a polynomial on a bounded box given by Formula (15) are not always exact. There is, however, a method that can be used to significantly improve the accuracy of the estimates for the minimum and maximum values of a multivariate polynomial  $p$  in a bounded box  $[\mathbf{a}, \mathbf{b}]$ . The basic idea is to subdivide  $[\mathbf{a}, \mathbf{b}]$  into two boxes by picking a variable  $x_j$ , where  $j < m$ , and consider the case where  $a_j \leq x_j \leq \frac{a_j+b_j}{2}$  separately from the case where  $\frac{a_j+b_j}{2} \leq x_j \leq b_j$ . This method can be used recursively to compute arbitrarily precise bounds of the minimum and maximum values of the polynomial on  $[\mathbf{a}, \mathbf{b}]$ .

An important feature of the subdivision method is that the Bernstein coefficients arising from the polynomial on the two subdivided intervals can be computed directly from the Bernstein coefficients of the original polynomial. Since the functions  $\lambda x: \frac{x}{2}$  and  $\lambda x: \frac{x+1}{2}$  are bijections from  $[0, 1]$  into  $[0, \frac{1}{2}]$  and  $[\frac{1}{2}, 1]$ , respectively, the Bernstein coefficients of a polynomial  $p$  on the boxes  $[\mathbf{0}, \mathbf{1} \text{ with } [j \leftarrow \frac{1}{2}]]$  and  $[\mathbf{0} \text{ with } [j \leftarrow \frac{1}{2}], \mathbf{1}]$  are the Bernstein coefficients of the polynomials  $p(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j}{2}])$  and  $p(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j+1}{2}])$ , respectively. The de Casteljau algorithm [4] is commonly used to compute the Bernstein forms of  $p(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j}{2}])$  and  $p(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j+1}{2}])$ . In this paper, a simpler variant of the de Casteljau algorithm is presented. In this variant, the Bernstein coefficients are computed by expanding the definitions of these polynomials.

**Proposition 9.** *Let  $p(\mathbf{x}) = \sum_{\mathbf{i} \leq \mathbf{n}} \hat{b}_{\mathbf{i}} B_{\mathbf{n},\mathbf{i}}(\mathbf{x})$  be an  $m$ -variable polynomial in Bernstein form. For all  $j < m$ ,  $p(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j}{2}]) = \sum_{\mathbf{k} \leq \mathbf{n}} \hat{b}_{\mathbf{k}}^L B_{\mathbf{n},\mathbf{k}}(\mathbf{x})$ , where*

$$\hat{b}_{\mathbf{k}}^L \equiv \sum_{r=0}^{k_j} \frac{1}{2^{k_j}} \binom{k_j}{r} \hat{b}_{\mathbf{k} \text{ with } [j \leftarrow r]},$$

and  $p(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j+1}{2}]) = \sum_{\mathbf{k} \leq \mathbf{n}} \hat{b}_{\mathbf{k}}^R B_{\mathbf{n},\mathbf{k}}(\mathbf{x})$ , where

$$\hat{b}_{\mathbf{k}}^R \equiv \sum_{r=0}^{n_j-k_j} \frac{1}{2^{n_j-k_j}} \binom{n_j-k_j}{r} \hat{b}_{\mathbf{k} \text{ with } [j \leftarrow n_j-r]}.$$

In PVS, functions `subdivl` and `subdivr` are defined that take as inputs a representation of an  $m$ -variable polynomial in Bernstein form  $p$  and an index  $j < m$ , and return the representations of the  $m$ -variable polynomials in Bernstein form

given in Proposition 9 for the polynomials  $p(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j}{2}])$  and  $p(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j+1}{2}])$ , respectively. The following proposition follows from Proposition 9 and Formula (13).

**Proposition 10.** *For any  $m$ -variable polynomial  $p$  and for all  $\mathbf{x} \in \mathbb{R}^m$ ,*

$$\begin{aligned} \text{eval}(\text{subdivl}(\text{tomultibern}(p), j)(\mathbf{x})) &= p(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j}{2}]), \\ \text{eval}(\text{subdivr}(\text{tomultibern}(p), j)(\mathbf{x})) &= p(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j+1}{2}]). \end{aligned}$$

The functions `subdivl` and `subdivr` can be used to improve the accuracy of the estimates for the minimum and maximum values of a polynomial in Bernstein form  $p$  on the unit box. This result is captured in the following proposition, the proof of which follows directly from Proposition 5 and Proposition 9.

**Proposition 11.** *Let  $p(\mathbf{x}) = \sum_{\mathbf{i} \leq \mathbf{n}} \hat{b}_{\mathbf{i}} B_{\mathbf{n}, \mathbf{i}}(\mathbf{x})$  be an  $m$ -variable polynomial in Bernstein form,  $K$  be a real number, and  $\Re$  be a real order in  $\{\leq, <, \geq, >\}$ . If  $\hat{b}_{\mathbf{i}}^L \Re K$  and  $\hat{b}_{\mathbf{i}}^R \Re K$ , for all  $\mathbf{i} \leq \mathbf{n}$ , then  $p(\mathbf{x}) \Re K$ , for all  $\mathbf{x} \in [0, 1]$ .*

The function `berncoeffsminmax`, when applied to `subdivl(tomultibern( $p$ ),  $j$ )` and `subdivr(tomultibern( $p$ ),  $j$ )`, returns elements of type `Outminmax` that represent range information for the polynomial  $p$  on the “left” and “right” (in the  $j$ -th variable) halves of the unit box, respectively. The function `combine` is used to aggregate the information from these elements of `Outminmax` into one element of `Outminmax` that represents range information for the polynomial  $p$  on the entire unit box  $[0, 1]$ . Given elements `omm1` and `omm2` of type `Outminmax`, `combine(omm1, omm2)` returns an element `omm` that satisfies

- `omm.lbmin` = `min(omm1.lbmin, omm2.lbmin)`,
- `omm.lbmax` = `min(omm1.lbmax, omm2.lbmax)`,
- `omm.lbvar` is `omm1.lbvar` if `omm.lbmax` = `omm1.lbmax`, otherwise `omm.lbvar` = `omm2.lbvar`,
- `omm.ubmin` = `max(omm1.ubmin, omm2.ubmin)`,
- `omm.ubmax` = `max(omm1.ubmax, omm2.ubmax)`, and
- `omm.ubvar` is `omm1.ubvar` if `omm.ubmax` = `omm1.ubmax`, otherwise `omm.ubvar` = `omm2.ubvar`.

Let  $\mathbf{p}$  be a polynomial in Bernstein form that represents the polynomial  $p$ , i.e.,  $\mathbf{p} = \text{tomultibern}(p)$ . It is not true that combining the outputs of  $\text{subdivl}(\mathbf{p}, j)$  and  $\text{subdivr}(\mathbf{p}, j)$  produces an element of `Outminmax` that satisfies the predicate *sound?*. The fields `lbvar` and `ubvar` in  $\text{omm}_l = \text{berncoeffsminmax}(\text{subdivl}(\mathbf{p}, j))$  and  $\text{omm}_r = \text{berncoeffsminmax}(\text{subdivr}(\mathbf{p}, j))$  are points in the unit box where the polynomials represented by  $\text{subdivl}(\mathbf{p}, j)$  and  $\text{subdivr}(\mathbf{p}, j)$  attain the values `lbmax` and `ubmin`, respectively. However, each of these polynomials represents  $p$  on only half of the unit box  $[0, 1]$ . Thus, the fields `omml.lbvar`, `omml.ubvar`, `ommr.lbvar`, and `ommr.ubvar` have to be translated from  $[0, 1]$  back to their respective half intervals for the predicate *sound?* to hold. This translation is accomplished through the functions `update` and `updateOutminmax`. The function `update` takes as inputs a tuple  $\mathbf{c} \in \mathbb{R}^m$ , a function  $u: \mathbb{R} \rightarrow \mathbb{R}$ , and a variable index  $j < m$ . It returns a tuple that is equal to  $\mathbf{c}$  in all components but the  $j$ -th component where it has the value  $u(c_j)$ , i.e.,

$$\text{update}(\mathbf{c}, u, j) \equiv \mathbf{c} \text{ with } [j \leftarrow u(c_j)].$$

The function `updateOutminmax` takes an element `omm` of type `Outminmax`, a function  $u: \mathbb{R} \rightarrow \mathbb{R}$ , and a variable index  $j < m$  as inputs. It updates the  $j$ -th components of the  $m$ -tuples `omm.lbvar` and `omm.ubvar` using the function  $u$ :

$$\begin{aligned} \text{updateOutminmax}(\text{omm}, u, j) \equiv \text{omm} \text{ with } [ & \text{lbvar} \leftarrow \text{update}(\text{omm.lbvar}, u, j), \\ & \text{ubvar} \leftarrow \text{update}(\text{omm.ubvar}, u, j) ]. \end{aligned}$$

**Proposition 12.** *For any  $m$ -variable polynomial  $p$ , variable index  $j < m$ , and elements  $\text{omm}_l, \text{omm}_r$  of type `Outminmax` such that*

$$\begin{aligned} \text{omm}_l &= \text{updateOutminmax}(\text{subdivl}(\text{tomultibern}(p), j), \lambda x: \frac{x}{2}, j), \\ \text{omm}_r &= \text{updateOutminmax}(\text{subdivr}(\text{tomultibern}(p), j), \lambda x: \frac{x+1}{2}, j), \end{aligned}$$

*$\text{sound?}(p, \text{combine}(\text{omm}_l, \text{omm}_r), \mathbf{0}, \mathbf{1})$  holds.*

## 5 The Maximum of Two Polynomials

If  $f$  and  $g$  are two functions from  $\mathbb{R}^m$  into  $\mathbb{R}$ , then the function  $\max(f, g)$  from  $\mathbb{R}^m$  into  $\mathbb{R}$  is defined by

$$\max(f, g)(\mathbf{x}) \equiv \max(f(\mathbf{x}), g(\mathbf{x})).$$

If the functions  $f$  and  $g$  are polynomials, then the function `berncoeffsminmax` discussed in §4.2 can be used to compute an element of `Outminmax` that satisfies the

predicate *sound?* on the unit box for the function  $\max(f, g)$ . This is accomplished through the function

$$\text{max\_bc\_minmax}(p, q) \equiv \text{combine\_max}(\text{berncoeffsminmax}(p), \text{berncoeffsminmax}(q)),$$

where  $p = \text{tomultibern}(f)$  and  $q = \text{tomultibern}(g)$ . In the definition of this function, *berncoeffsminmax* is used to compute two elements of *Outminmax* that contain range information for  $f$  and  $g$  on  $[0, 1]$ . The function *combine\_max* then computes the worst case scenario for the range of the function  $\max(f, g)$  on  $[0, 1]$ , and it is defined as follows.

$$\text{combine\_max}(\text{omm}_1, \text{omm}_2) \equiv \begin{cases} \text{omm}_1 & \text{if } \text{omm}_1.\text{lbmin} \geq \text{omm}_2.\text{ubmax}, \\ \text{omm}_2 & \text{if } \text{omm}_2.\text{lbmin} > \text{omm}_1.\text{ubmax}, \\ \text{omm} & \text{otherwise,} \end{cases}$$

where  $\text{omm}.\text{lbmin} = \min(\text{omm}_1.\text{lbmin}, \text{omm}_2.\text{lbmin})$ ,  $\text{omm}.\text{lbmax} = 0$ ,  $\text{omm}.\text{ubmin} = 0$ ,  $\text{omm}.\text{ubmax} = \max(\text{omm}_1.\text{ubmax}, \text{omm}_2.\text{ubmax})$ , and both  $\text{omm}.\text{lbvar}$  and  $\text{omm}.\text{ubvar}$  are set to empty.

**Proposition 13.** *For any  $m$ -variable polynomials  $p$  and  $q$ ,*

$$\text{sound?}(\max(p, q), \text{max\_bc\_minmax}(p, q), 0, 1)$$

*holds, where  $p = \text{tomultibern}(p)$  and  $q = \text{tomultibern}(q)$ .*

Proposition 13 states that the function *max\_bc\_minmax* computes bounds on the minimum and maximum values of the function  $\max(p, q)$  in the unit box. Although these bounds are correct, they may not be precise enough to determine whether a relation such as  $\max(p, q) \geq K$  is satisfied for a given real number  $K$ . In this case, Proposition 11 can be recursively applied to subdivide the unit box into smaller intervals and compute estimates that are precise to any required approximation.

## 5.1 A Procedure For Checking $\max(p, q) \geq K$ in $[0, 1]$

The following procedure can be used to determine whether the maximum  $\max(p, q)$  of two  $m$ -variable polynomials  $p$  and  $q$  always takes a value of at least  $K$  on the unit box  $[0, 1]$ .

1. Compute  $\text{omm} = \text{max\_bc\_minmax}(\text{tomultibern}(p), \text{tomultibern}(q))$ .
2. If  $\text{omm}.\text{lbmin} \geq K$ , then the inequality  $\max(p, q)(\mathbf{x}) \geq K$  holds for all  $\mathbf{x} \in [0, 1]$ . The procedure exits with success.

3. If  $\text{omm.lbmax} < K$ , then the inequality  $\max(p, q)(\mathbf{c}) < K$  holds for  $\mathbf{c} = \text{omm.lbvar} \in [0, 1]$ . The procedure exits with failure, with  $\mathbf{c}$  as counterexample.
4. Otherwise, choose any  $0 \leq j < m$  and recursively apply this procedure to determine whether

$$(a) \max(p, q)(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j}{2}]) \geq K \text{ and}$$

$$(b) \max(p, q)(\mathbf{x} \text{ with } [j \leftarrow \frac{x_j+1}{2}]) \geq K,$$

for all  $\mathbf{x} \in [0, 1]$ .

- If both Step 4a and Step 4b exit with success, then, by Proposition 11, the inequality  $\max(p, q)(\mathbf{x}) \geq K$  holds for all  $\mathbf{x} \in [0, 1]$ . The procedure exits with success.
- If Step 4a exits with failure with counterexample  $\mathbf{c}$ , then the inequality  $\max(p, q)(\mathbf{c}_l) < K$  holds, where  $\mathbf{c}_l = \mathbf{c} \text{ with } [j \leftarrow \frac{x_j}{2}] \in [0, 1]$ . The process exits with failure, with  $\mathbf{c}_l$  as counterexample.
- If Step 4b exits with failure with counterexample  $\mathbf{c}$ , then the inequality  $\max(p, q)(\mathbf{c}_r) < K$  holds, where  $\mathbf{c}_r = \mathbf{c} \text{ with } [j \leftarrow \frac{x_j+1}{2}] \in [0, 1]$ . The process exits with failure, with  $\mathbf{c}_r$  as counterexample.

It should be noted that the procedure above does not necessarily terminate. However, at each recursive step, the interval  $[\text{omm.lbmin}, \text{omm.lbmax}]$ , which contains the minimum value of  $\max(p, q)$  over the unit box  $[0, 1]$ , gets smaller.

Given  $m$ -variable polynomials  $p$  and  $q$  and a real number  $K$ , the procedure above can also be used to determine whether the inequality  $\max(p, q)(\mathbf{x}) \geq K$  holds for all  $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ . In this case, the procedure is used with the  $m$ -variable polynomials  $p_{[\mathbf{a}, \mathbf{b}]} = \text{translate}(\mathbf{a}, \mathbf{b}, p)$  and  $q_{[\mathbf{a}, \mathbf{b}]} = \text{translate}(\mathbf{a}, \mathbf{b}, q)$ . If the procedure exits with success, then the inequality  $\max(p, q)(\mathbf{x}) \geq K$  holds for all  $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ . If the procedure exits with failure with counterexample  $\mathbf{c}$ , the inequality  $\max(p, q)(\mathbf{c}') < K$  holds, where  $\mathbf{c}' = \sigma_{[\mathbf{a}, \mathbf{b}]}(\mathbf{c})$ .

The complexity of the procedure is exponential in the number of variables. However, some heuristics can be used to greatly speed this procedure [15, 19, 22]. For instance, Step 4 involves applying the procedure recursively on the left and right hand sides of the unit box. However, if the procedure is run for the left (resp. right) side of the unit box first, in some cases, it may not be necessary to run it for the right (resp. left) hand side at all. An example of this is when the recursive call on the right (resp. left) hand side exits with failure with a counterexample  $\mathbf{c}$ . In this case,  $\mathbf{c}$  is also a counterexample for the larger box, so running the procedure on the left (resp. right) hand side of the box is not necessary. There

are also heuristics for the selection of the variable  $j$  in Step 4 and of the order in which Step 4a and Step 4b are performed. These heuristics improve the efficiency of the procedure by pruning the execution tree generated by the recursive calls in Step 4.

## 5.2 The Algorithm `bernMinmax`

This section describes a formally verified algorithm, which is based on the procedure presented in §5.1, that computes range information for the minimum and maximum values of a function defined as the maximum of two polynomials.

The function `bernMinmax`, defined in Figure 7, has as inputs representations  $\mathbf{p}$  and  $\mathbf{q}$  of  $m$ -variable polynomials in Bernstein form  $p$  and  $q$ , respectively, a maximum recursion depth  $\mathbf{N} \in \mathbb{N}$ , and the current recursion depth  $\mathbf{i} \leq \mathbf{N}$ . Additional inputs include a function `varsel` that selects the variable on which to subdivide at each iteration and which direction to explore first, predicates `localex` and `globalex` on the output type that cause the algorithm to exit locally and globally, respectively, and a parameter `omm` of the same type as the output value. These additional inputs allow for the use of heuristics based on a particular variable selection method, direction of recursion, and strategy for early termination. They are described in §5.4.

The function `bernMinmax` returns a record of type `Outminmax`. Assuming that  $\mathbf{bmm} = \text{max\_bc\_minmax}(\mathbf{p}, \mathbf{q})$  as in Line 1 of this function, if the condition in Line 3 is true, then the function returns  $\mathbf{bmm}$ . In this case,  $\mathbf{bmm}$  will satisfy either `localex` or `globalex`, except when the maximum depth has been reached, i.e.,  $\mathbf{i} = \mathbf{N}$ , or when the execution tree is pruned by the condition `between(omm, bmm)`.

If the condition in Line 3 is false, then the function `varsel` is used to select a variable to subdivide and a direction (left or right) for each one of the polynomials. It is important to note that the function `varsel` is an input to `bernMinmax`, so it can handle any subdivision scheme. Next, the domain subdivision functions `subdivl` and `subdivr` presented in §4.3, are used to subdivide the unit box  $[0, 1]$  into smaller sub-boxes. At each subdivision, the function `max_bc_minmax` is recursively called to compute an element of `Outminmax` that stores information about the ranges of the polynomials on the given sub-box. Using this function, two elements  $\mathbf{bmm}_1$  and  $\mathbf{bmm}_2$  of `Outminmax` are produced; one representing range information for the box  $[0, 1 \text{ with } [j \leftarrow \frac{1}{2}]]$  and the other representing range information for the box  $[0 \text{ with } [j \leftarrow \frac{1}{2}], 1]$ . The algorithm effectively stops when the predicate `globalex` is satisfied after the first of the two recursive calls of the function `bernMinmax`. This is accomplished by the condition in Line 17. Since the points represented by `lbvar` and `ubvar` are computed in a unit box, they must be translated back to the half boxes from the full boxes in the function, by using the

```

01 : bernMinmax(p, q, N, i, varsel, localex, globalex, omm): Outminmax ≡
02 :   let bmm = max_bc_minmax(p, q) in
03 :     if i = N or localex(bmm) or (i > 0 and between(omm, bmm)) or
04 :       globalex(bmm) then bmm
05 :     else
06 :       let (leftp, jp) = varsel(p, i),
07 :         (leftq, jq) = varsel(q, i),
08 :         (left, j) = if mod(i, 2) = 0 then (leftp, jp)
09 :           else (leftq, jq) endif,
10 :         (pl, pr) = (subdivl(p, j), subdivr(p, j)),
11 :         (ql, qr) = (subdivl(q, j), subdivr(q, j)),
12 :         (p1, p2) = if left then (pl, pr) else (pr, pl) endif,
13 :         (q1, q2) = if left then (ql, qr) else (qr, ql) endif,
14 :         σ = if left then λx: x/2 else λx: (x + 1)/2 endif,
15 :         omm = if i > 0 then combine(omm, bmm) else bmm endif,
16 :         bmm1 = bernMinmax(p1, q1, N, i + 1, varsel, omm) in
17 :       if globalex(bmm1) then
18 :         combine(updateOutminmax(bmm1, σ, j), bmm)
19 :       else
20 :         let omm = combine(omm, bmm1),
21 :           bmm2 = bernMinmax(p2, q2, N, i + 1, varsel, omm),
22 :           bmmleft = if left then bmm1 else bmm2 endif,
23 :           bmmright = if left then bmm2 else bmm1 endif in
24 :         combine(updateOutminmax(bmmleft, λx: x/2, j),
25 :           updateOutminmax(bmmright, λx: (x + 1)/2, j))
26 :       endif
27 :     endif

```

Figure 7: The function `bernMinmax`



function `updateOutminmax`.

If the condition in Line 17 is false, the two elements of type `Outminmax` resulting from applying `updateOutminmax` to `bmm1` and `bmm2` are combined into a new element of type `Outminmax` that represents the range information of the function  $\max(p, q)$  over the unit box.

The correctness property of the function `bernMinmax` states that it computes an element of type `Outminmax` that bounds the range of the function  $\max(p, q)$  over the unit box. The following theorem has been proved in PVS by induction on the structure of the definition of `bernMinmax`. Proposition 13 is used to prove the base case. The inductive case is discharged by Proposition 12 in §4.3.

**Theorem 2.** *For all  $m$ -variable polynomials  $p, q: \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $N \in \mathbb{N}$ ,  $i \in \mathbb{N}$ , with  $i \leq N$ ,  $\text{varsel}: \mathbb{N} \rightarrow \text{boolean} \times \mathbb{N}_{<m}$ ,  $\text{localex}, \text{globalex}: \text{Outminmax} \rightarrow \text{boolean}$ , and  $\text{omm} \in \text{Outminmax}$ , if  $\text{bmm} \in \text{Outminmax}$  is given by*

$$\text{bmm} = \text{bernMinmax}(\text{tomultibern}(p), \text{tomultibern}(q), N, i, \text{varsel}, \text{omm}),$$

*then*

1.  $\max(p, q)(\text{bmm.lbvar}) = \text{bmm.lbmax}$ ,
2.  $\max(p, q)(\text{bmm.ubvar}) = \text{bmm.ubmin}$ , and
3.  $\text{bmm.lbmin} \leq \max(p, q)(x) \leq \text{bmm.ubmax}$ , for all  $x \in [0, 1]$ .

It is noted that Theorem 2 holds for all possible values of the input parameters `varsel`, `localex`, `globalex`, and `omm`. These parameters are added for practical and efficiency reasons. They are explained in §5.4.

### 5.3 Function `polyMinmax`

The function `polyMinmax`, defined in Figure 8, computes range information for the maximum of two polynomials on an arbitrary box  $[a, b]$ . The algorithm works in four steps:

1. Convert the polynomials from the box  $[a, b]$  to the unit box  $[0, 1]$  using the function `translate` defined by Formula (4) in §4.
2. Compute the Bernstein form of the translated polynomial using the function `tomultibern` from §4.1.
3. Apply `bernMinmax` to compute an element `bmm` of `Outminmax` that gives range information for the maximum of these two polynomials in Bernstein form on the unit box.

```

polyMinmax( $p, q, \mathbf{a}, \mathbf{b}, N, \text{varsel}, \text{localex}, \text{globalex}$ ): Outminmax  $\equiv$ 
  let  $p = \text{tomultibern}(\text{translate}(p, \mathbf{a}, \mathbf{b}))$ ,
       $q = \text{tomultibern}(\text{translate}(q, \mathbf{a}, \mathbf{b}))$ ,
       $\text{omm} = \text{bernMinmax}(p, q, N, 0, \text{varsel}, \text{localex}, \text{globalex}, \text{emptymm})$ 
  in
    denorm_omm( $\text{omm}, \mathbf{a}, \mathbf{b}$ )

```

Figure 8: The function polyMinmax

4. Translate  $\text{omm}$  from  $[0, 1]$  back to  $[\mathbf{a}, \mathbf{b}]$  linearly using the function  $\text{denorm\_omm}$  defined in §4.2.

The constant element  $\text{emptymm}$  of type  $\text{Outminmax}$  is defined such that all the numerical fields are 0 and the  $m$ -tuples are empty. The following correctness property of the function  $\text{polyMinmax}$  has been proved in PVS.

**Theorem 3.** *For all  $m$ -variable polynomials  $p, q: \mathbb{R}^m \rightarrow \mathbb{R}$  in standard form,  $N \in \mathbb{N}$ ,  $\text{varsel}: \mathbb{N} \rightarrow \text{boolean} \times \mathbb{N}_{<m}$ , and  $\text{localex}, \text{globalex}: \text{Outminmax} \rightarrow \text{boolean}$ , if  $\text{omm} \in \text{Outminmax}$  is given by*

$$\text{omm} = \text{polyMinmax}(p, q, \mathbf{a}, \mathbf{b}, N, \text{varsel}, \text{localex}, \text{globalex}),$$

then

- $\max(p, q)(\text{omm.lbvar}) = \text{omm.lbmax}$ ,  $\max(p, q)(\text{omm.ubvar}) = \text{omm.ubmin}$ , and
- $\text{omm.lbmin} \leq \max(p, q)(\mathbf{x}) \leq \text{omm.ubmax}$ , for all  $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ .

According to Theorem 3, the range information computed by  $\text{polyMinmax}$  can be used to check universally quantified propositions of the form

$$\forall \mathbf{x} \in [\mathbf{a}, \mathbf{b}]: \max(p, q)(\mathbf{x}) \mathfrak{R} K,$$

for any given  $m$ -variable polynomials  $p, q$ , constant value  $K$ , and real order relation  $\mathfrak{R} \in \{\geq, >, \leq, <\}$ . If  $\mathfrak{R}$  is one of  $\geq$  or  $>$ , it suffices to check whether  $\text{omm.lbmin} \mathfrak{R} K$ . If  $\mathfrak{R}$  is one of  $\leq$  or  $<$ , it suffices to check whether  $\text{omm.ubmax} \mathfrak{R} K$ . Furthermore, since an existential proposition of the form “ $\exists \mathbf{c} \in [\mathbf{a}, \mathbf{b}]: \max(p, q)(\mathbf{c}) \mathfrak{R} K$ ” is equivalent to the negation of the universal proposition “ $\forall \mathbf{x} \in [\mathbf{a}, \mathbf{b}]: \max(p, q)(\mathbf{x}) \neg \mathfrak{R} K$ ”, the former type of existentially qualified propositions can also be checked using the algorithm  $\text{polyMinmax}$ . In this case, the witness  $\mathbf{c}$  is either  $\text{omm.lbvar}$  or  $\text{omm.ubvar}$  depending on  $\mathfrak{R}$ .

## 5.4 Parameters `varsel`, `omm`, `localex`, and `globalex`

The parameter `varsel` is used to determine two things: (1) Which variable to subdivide at each recursive step, and (2) Whether to compute bounds to the left or the right first in that variable. The algorithm takes as inputs a representation `p` of an  $m$ -variable polynomial in Bernstein form  $p$  and a natural number `i`. It returns a pair `(left, j)`, where `left` is a Boolean value and  $j < m$ . The value `left` being `true` means that the given variable should be subdivided to the left first, and  $j$  is a natural number representing the index of the variable to be subdivided. The most basic example of such a function is given by `varsel(p, i) = (true, mod(m, i))`, which alternates the variables and always computes range information on the left interval first. However, as noted in [15] and [19], there are much more efficient methods for choosing these variables and directions, including several based on derivatives. The function `varsel` is an input to the algorithm in PVS, so it can facilitate any subdivision scheme. One method that has been implemented in PVS is called `MaxVarMinDir`. This method chooses the variable for which the range between the first and last Bernstein coefficients, when all other variables are held constant, is greatest. In the algorithm `bernMinmax`, the function `varsel` is called on both `p` and `q`, and the answer that is used alternates between these answers.

The parameter `omm` is used for caching the current output of the algorithm. The function `between` is defined as follows

$$\begin{aligned} \text{between}(\text{omm}, \text{bmm}) \equiv & \text{omm.lbmax} \leq \text{bmm.lbmin} \text{ and} \\ & \text{bmm.ubmax} \leq \text{omm.ubmin}. \end{aligned}$$

It tests whether the output `bmm` at the current recursive step can contribute anything to the final output of the function once it is combined. At a given recursive step in the algorithm, if `between(omm, bmm)` returns `true`, then the output `bmm` of the current recursive step will not contribute to the overall output of the function since `between(omm, bmm)` implies that `combine(omm, bmm) = omm`.

The predicates `localex` and `globalex` are used to prune the executing tree depending on particular objectives. The predicate `localex` will be used to exit the algorithm locally and continue to the next recursive step. While both of these predicates are used in the algorithm to simply break recursion locally, the predicate `globalex` will be chosen so that if recursion breaks because `globalex` returns true, then every recursion above will also break, effectively resulting in a global exit from the algorithm. For instance, to check propositions of the form “ $\forall x \in [a, b]: \max(p, q)(x) \geq K$ ”, `localex` and `globalex` can be defined as follows.

$$\begin{aligned} \text{localex}(\text{omm}) \equiv & \text{omm.lbmin} \geq K, \\ \text{globalex}(\text{omm}) \equiv & \text{omm.lbmax} < K. \end{aligned}$$

In this case, if a box satisfies the inequality, there is no need to subdivide the box. On the other hand, if the negated inequality is satisfied by a point in the box, the algorithm must exit since a counterexample to the universally quantified inequality has been found.

## 6 Conflict Detection for Arbitrary Trajectories

It is noted in §3.3 that conflict detection between two trajectories  $P_o$  and  $P_i$  over a box  $[\mathbf{a}, \mathbf{b}]$  can be reduced to determining whether the function  $\mathbf{sqdist}_{D,H,P_o,P_i}$  ever takes a value less than 1 on  $[\mathbf{a}, \mathbf{b}]$ . The function  $\mathbf{sqdist}_{D,H,P_o,P_i}$  is defined as the maximum of the functions  $f(\mathbf{x}) = \frac{w_x^2}{H^2}$  and  $g(\mathbf{x}) = \frac{w_x^2 + w_y^2}{D^2}$ , where  $\mathbf{w} = P_o(\mathbf{x}) - P_i(\mathbf{x})$ . The numerical constants  $D$  and  $H$  represent the minimum horizontal and vertical separation between the aircraft, respectively.

If  $P_o$  and  $P_i$  are polynomial functions, the functions  $f$  and  $g$  are also polynomials, and therefore the function  $\mathbf{sqdist}_{D,H,P_o,P_i}$  is the maximum of two polynomials. The next proposition follows from Theorem 1 in §3.3 and the definition of the predicate *sound?*.

**Proposition 14.** *If  $\text{sound?}(\mathbf{sqdist}_{D,H,P_o,P_i}, \text{omm}, \mathbf{a}, \mathbf{b})$  holds, then*

- *If  $\text{omm.lbmin} \geq 1$ , then  $\text{conflict?}(\mathbf{a}, \mathbf{b}, P_o, P_i)$  does not hold.*
- *If  $\text{omm.lbmax} < 1$ , then  $\text{conflict?}(\mathbf{a}, \mathbf{b}, P_o, P_i)$  holds, and conflict is attained at the point  $\text{lbvar}$ , which is an element of  $[\mathbf{a}, \mathbf{b}]$ .*

According to Proposition 14 and Theorem 3 in §5, conflict between trajectories  $P_o$  and  $P_o$  can be determined by using the algorithm `polyMinmax` defined in §5.3. For this purpose, the local exit and global exit predicates used in the algorithm `bernMinmax` can be defined as follows.

$$\begin{aligned} \text{cd\_localexit}(\text{omm}) &\equiv \text{omm.lbmin} \geq 1, \\ \text{cd\_globalexit}(\text{omm}) &\equiv \text{omm.lbmax} < 1. \end{aligned}$$

Alternatively, if the trajectories  $P_o$  and  $P_i$  are not defined by polynomials but are approximated by polynomial trajectories  $P'_o$  and  $P'_i$ , the discussion in Section 3.3, including Proposition 1, implies that conflict information for  $P_o$  and  $P_i$  can be computed by considering the function  $\mathbf{sqdist}_{D,H,P'_o,P'_i}$ , which is the maximum of two polynomials. In this case, the local exit and global exit predicates are parametric on  $D$ ,  $H$ ,  $\varepsilon_D$ , and  $\varepsilon_H$ , where  $\varepsilon_D$  and  $\varepsilon_H$  represent desired precisions

in the horizontal and vertical dimensions as explained in §3.3. These parametric predicates are defined as follows.

$$\begin{aligned}\text{cd\_localexit}_{D,H,\varepsilon_D,\varepsilon_H}(\text{omm}) &\equiv \text{omm.lbmin} \geq 1 + \delta_+(D, H, \varepsilon_D, \varepsilon_H), \\ \text{cd\_globalexit}_{D,H,\varepsilon_D,\varepsilon_H}(\text{omm}) &\equiv \text{omm.lbmax} < 1 - \delta_-(D, H, \varepsilon_D, \varepsilon_H),\end{aligned}$$

where  $\delta_+(D, H, \varepsilon_D, \varepsilon_H)$  and  $\delta_-(D, H, \varepsilon_D, \varepsilon_H)$  are defined by Formula (8) in §3.3.

Based on these results, this section presents a formally verified conflict detection algorithm for arbitrary trajectories.

## 6.1 Verified Conflict Detection Algorithm

Suppose that  $P_o$  and  $P_i$  are polynomial trajectories from  $\mathbb{R}^m$  into  $\mathbb{R}^3$ . The algorithm `cd_poly`, in Figure 9, can be used to check whether these two trajectories are in conflict on a given box  $[\mathbf{a}, \mathbf{b}]$ , i.e., whether  $\text{conflict?}(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i)$  holds. The parametric function  $\text{cd\_poly}_{N,\varepsilon_D,\varepsilon_H}$  is a conflict detection algorithm as defined in §3.2. It has as inputs positive real numbers  $D, H$ , a box  $[\mathbf{a}, \mathbf{b}]$ , and polynomial trajectories  $P_o, P_i$ . It returns an element of type `CDOutcome` with the values `Unknown`, `NoConflict`, or `LossAt(c)`, where  $\mathbf{c} \in [\mathbf{a}, \mathbf{b}]$  is a point at which the trajectories loss separation. The parameter  $N \in \mathbb{N}$  represents a maximum depth for the bounding algorithm `polyMinmax` presented in §5.3. The parameters  $\varepsilon_D$  and  $\varepsilon_H$  are usually 0, except when  $P_o$  and  $P_i$  are polynomial approximations of non-polynomial trajectories in which case they are small nonnegative real numbers.

Theorem 4 states that the algorithm  $\text{cd\_poly}_{N,\varepsilon_D,\varepsilon_H}$  is correct, as defined in §3.2, when  $\varepsilon_D = \varepsilon_H = 0$ .

**Theorem 4** (Correctness for Polynomial Trajectories). *For all polynomial trajectories  $P_o$  and  $P_i$  from  $\mathbb{R}^m$  into  $\mathbb{R}^3$ , boxes  $[\mathbf{a}, \mathbf{b}] \in \mathbb{R}^m$ , and depths  $N \in \mathbb{N}$ ,*

1.  $\text{cd\_poly}_{N,0,0}(D, H, P_o, P_i, \mathbf{a}, \mathbf{b}) = \text{NoConflict}$  implies

$$\neg \text{conflict?}(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i).$$

2.  $\text{cd\_poly}_{N,0,0}(D, H, P_o, P_i, \mathbf{a}, \mathbf{b}) = \text{LossAt}(\mathbf{c})$  implies

$$\text{los?}(D, H, P_o(\mathbf{c}), P_i(\mathbf{c}))$$

and hence  $\neg \text{conflict?}(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i)$  holds.

Theorem 4 is a particular case of a more general theorem that takes the parameters  $\varepsilon_D$  and  $\varepsilon_H$  into account. Theorem 5 shows that the algorithm `cd_poly` can also be used to compute conflict information for arbitrary trajectories, assuming that polynomial approximations are known.

```

cd_polyN,εD,εH(D, H, Po, Pi, a, b) : CDOutcome ≡
  let  p = λx: (Po(x)z - Pi(x)z)2/H2,
      q = λx: (Po(x)(x,y) - Pi(x)(x,y))2/D2,
      omm = polyMinmax(p, q, a, b, N, MaxVarMinDir,
        cd_localexitD,H,εD,εH,
        cd_globalexitD,H,εD,εH) in
  if cd_localexitD,H,εD,εH(omm) then
    IsFalse
  elsif cd_globalexitD,H,εD,εH(omm) then
    Conflict(omm.lbvar)
  else
    Unknown
  endif

```

Figure 9: The function `cd_poly`

**Theorem 5** (Correctness for Arbitrary Trajectories). *For all aircraft trajectories  $P_o, P_i: \mathbb{R}^m \rightarrow \mathbb{R}^3$ , polynomial trajectories  $P'_o, P'_i: \mathbb{R}^m \rightarrow \mathbb{R}^3$ , nonnegative real numbers  $\varepsilon_D = \varepsilon_{oD} + \varepsilon_{iD}$  and  $\varepsilon_H = \varepsilon_{oH} + \varepsilon_{iH}$ , boxes  $[\mathbf{a}, \mathbf{b}] \in \mathbb{R}^m$ , and depths  $N \in \mathbb{N}$ , such that  $P'_o$  and  $P'_i$  are polynomial approximations of  $P_o$  and  $P_i$ , respectively, i.e.,  $\text{approx}?( \mathbf{a}, \mathbf{b}, \varepsilon_{oD}, \varepsilon_{oH}, P_o, P'_o )$  and  $\text{approx}?( \mathbf{a}, \mathbf{b}, \varepsilon_{iD}, \varepsilon_{iH}, P_i, P'_i )$  hold,*

1.  $\text{cd\_poly}_{N, \varepsilon_D, \varepsilon_H}(D, H, P'_o, P'_i, \mathbf{a}, \mathbf{b}) = \text{NoConflict}$  implies

$$\neg \text{conflict}?(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i).$$

2.  $\text{cd\_poly}_{N, \varepsilon_D, \varepsilon_H}(D, H, P'_o, P'_i, \mathbf{a}, \mathbf{b}) = \text{LossAt}(\mathbf{c})$  implies

$$\text{los}?(D, H, P_o(\mathbf{c}), P_i(\mathbf{c}))$$

and hence  $\neg \text{conflict}?(D, H, \mathbf{a}, \mathbf{b}, P_o, P_i)$  holds.

Notice that the applications of `cd_poly` in Theorem 5 involve the polynomial trajectories  $P'_o$  and  $P'_i$ , whereas the conclusions involve  $P_o$  and  $P_i$ . Theorem 4 and Theorem 5 have both been formally proved in PVS. Theorem 4 follows trivially from Theorem 5 by setting  $P_o = P_o$ ,  $P_i = P_i$ ,  $\varepsilon_D = 0$ , and  $\varepsilon_H = 0$ . The proof of Theorem 5 follows from Theorem 3 in §5.3 and Proposition 1 in §3.3.

Theorem 4 and Theorem 5 are proved in PVS and they are available as part of the formal development at <http://shemesh.larc.nasa.gov/people/cam/Bernstein>.

## 6.2 An Example with Polynomial Trajectories

Consider the following two polynomial trajectories of type  $\mathbb{R} \rightarrow \mathbb{R}^3$ , meaning that they only depend on the time parameter and do not involve uncertainties.

$$\begin{aligned} P_o(t) &= (-3.2484 + 270.7t + 433.12t^2 - 324.83999t^3, \\ &\quad 15.1592 + 108.28t + 121.2736t^2 - 649.67999t^3, \\ &\quad 38980.8 + 5414.0t - 21656.0t^2 + 32484.0t^3) \\ P_i(t) &= (1.0828 - 135.35t + 234.9676t^2 + 3248.4t^3, \\ &\quad 18.40759 - 230.6364t - 121.2736t^2 - 649.67999t^3, \\ &\quad 40280.15999 - 10828.0t + 24061.9816t^2 - 32484.0t^3) \end{aligned}$$

These trajectories represent the 3D Euclidean positions of the ownship and intruder aircraft, respectively.

The unit of time for these trajectories is *hours* (**hr**), the unit of horizontal position is *nautical miles* (**nm**), and the unit of vertical position is *feet* (**ft**). The minimum separation standard is 5**nm** horizontally and 1000**ft** vertically, i.e.,  $D = 5$  and  $H = 1000$ . The lookahead time is assumed to be 3 minutes, i.e.,  $T = \frac{1}{20}$ . Since the only variable of the polynomial is time, the box of interest for conflict detection is just the interval  $[0, T]$ .

At time  $t = 0$  hours (current time), the positions of the ownship and intruder aircraft are  $(-3.2484, 15.1592, 38980.8)$  and  $(1.0828, 18.40759, 40280.15999)$ , respectively. At this time, the aircraft are approximately 5.414**nm** apart horizontally, and approximately 1299.36**ft** apart vertically. Thus, given the minimum separation standard, the aircraft are not currently in loss of separation. However, the algorithm `cd.poly` predicts that the aircraft are in conflict. That is,

$$\text{cd.poly}_{N,0,0}(D, H, P_o, P_i) = \text{LossAt}(5105/262144), \quad (16)$$

where  $N = 1000$ . Theorem 4 states that these trajectories will be in loss of separation at time  $t = \frac{5105}{262144}$ , or in about 70 seconds. Hence,  $\text{conflict?}(D, H, 0, T, P_o, P_i)$  holds. Indeed, at this time, the aircraft are approximately 4.999**nm** apart horizontally and  $-999.92$ **ft** vertically.

It is important to note that these results, which have been proved in PVS, are not subject to computation errors. For instance, the evaluation of `cd.poly` in Formula (16) is mathematically equal to `Conflict(5105/262144)`. Furthermore,

at time  $t = \frac{5105}{262144}$  the horizontal between the aircraft is strictly less than 5 horizontally and strictly less than 1000 vertically.

### 6.3 Conflict Detection with Turning Trajectories

The trajectory of an aircraft in a steady turn, without considering uncertainties, is described by the parametric function  $\text{turn}_{\mathbf{s}, R, \alpha, \omega, v_z} : \mathbb{R} \rightarrow \mathbb{R}$ , with parameters  $\mathbf{s}$ ,  $R$ ,  $\alpha$ ,  $\omega$ , and  $v_z$ , defined by Formula 3 in §3.1. Even though the function  $\text{turn}$  is not defined by polynomials, it can be approximated by polynomials. For any given natural number  $n > 0$ , the following polynomial trajectory approximates the trajectory  $\text{turn}_{\mathbf{s}, R, \alpha, \omega, v_z}$ .

$$\begin{aligned} \text{turnpoly}_{\mathbf{s}, R, \alpha, \omega, v_z, n}(t) = \mathbf{s} + & \left( \sum_{i=0}^n R \frac{(-1)^i}{(2i+1)!} (\alpha + t\omega)^{2i+1}, \right. \\ & \sum_{i=0}^n R \frac{(-1)^i}{(2i)!} (\alpha + t\omega)^{2i}, \\ & \left. t v_z \right). \end{aligned} \quad (17)$$

The summations in the first and second components of this trajectory represent Taylor series expansions of  $R \sin(\alpha + t \cdot \omega)$  and  $R \cos(\alpha + t \cdot \omega)$ , respectively. The next lemma gives a bound for the error of this approximating trajectory, in terms of the predicate *approx?* defined in Section 3.3.

**Lemma 2.** *For any parameters  $\mathbf{s}$ ,  $\alpha$ ,  $R$ ,  $\omega$ ,  $v_z$ , and natural numbers  $n$ , the polynomial trajectory  $\text{turnpoly}_{\mathbf{s}, \alpha, R, \omega, v_z, n}$  approximates  $\text{turn}_{\mathbf{s}, \alpha, \omega, R, v_z}$  over the time interval between 0 and  $T$ , i.e.,*

$$\text{approx?}(0, T, \varepsilon_1 + \varepsilon_2, 0, \text{turn}_{\mathbf{s}, \alpha, \omega, R, v_z}, \text{turnpoly}_{\mathbf{s}, \alpha, R, \omega, v_z, n})$$

holds, where

$$\begin{aligned} \beta &\equiv \max(|\alpha|, |\alpha + T \cdot \omega|), \\ \varepsilon_1 &\equiv R \frac{\beta^{3+2n}}{(3+2n)!}, \\ \varepsilon_2 &\equiv R \frac{\beta^{2+2n}}{(2+2n)!}. \end{aligned}$$

It is easy to see that  $\varepsilon_1$  and  $\varepsilon_2$  in Lemma 2 converge to zero as  $n$  approaches infinity. While this gives a bound for the error of the approximation, it depends on  $n$ . Thus, it is helpful to first specify the desired precision  $\delta_D$  and then compute a number  $n$  such that  $\varepsilon_1 + \varepsilon_2 < \delta_D$ . This can be accomplished through a function



named `exp_term_num` that takes as inputs a nonnegative real number  $x$  and a positive precision  $\varepsilon > 0$ . It returns a natural number  $j > x + 1$  with the property that  $\frac{x^k}{k!} < \varepsilon$  for all  $k \geq 2j$ . The existence of such a function follows from the fact that  $\frac{x^k}{k!}$  converges to zero as  $k$  approaches infinity. There are many ways to define such a function, and in the PVS development it is defined recursively.

According to Lemma 2 and Theorem 5, the function `turnpoly` can be used to deduce conflict information for trajectories defined by `turn`. This is stated explicitly in the following theorems, which consider the case where both aircraft are turning and where only one aircraft is turning. These theorems have been formally proved in PVS.

**Theorem 6** (Both Aircraft Are Turning). *For all  $P_o, P'_o, P_i, P'_i, \mathbf{s}_o, \mathbf{s}_i, \alpha_o, \alpha_i, \omega_o, \omega_i, R_o, R_i, v_{oz}, v_{iz}, \beta_o, \beta_i, n, \mathbf{cd\_outcome}, N, D, H$ , and  $T$ , if*

- $P_o = \mathbf{turn}_{\mathbf{s}_o, \alpha_o, \omega_o, R_o, v_{oz}},$
- $P_i = \mathbf{turn}_{\mathbf{s}_i, \alpha_i, \omega_i, R_i, v_{iz}},$
- $\beta_o = \max(|\alpha_o|, |\alpha_o + T \cdot \omega_o|),$
- $\beta_i = \max(|\alpha_i|, |\alpha_i + T \cdot \omega_i|),$
- $n = \mathbf{exp\_term\_num}(\varepsilon_D / (4 \max(R_o, R_i)), \max(\beta_o, \beta_i)),$
- $P'_o = \mathbf{turnpoly}_{\mathbf{s}_o, \alpha_o, R_o, \omega_o, v_{oz}, n},$
- $P'_i = \mathbf{turnpoly}_{\mathbf{s}_i, \alpha_i, R_i, \omega_i, v_{iz}, n},$  and
- $\mathbf{cd\_outcome} = \mathbf{cd\_poly}(D, H, P'_o, P'_i, 0, T, N, \varepsilon_D, 0),$

then

1.  $\mathbf{cd\_outcome} = \mathbf{NoConflict}$  implies  $\neg \mathbf{conflict?}(D, H, 0, T, P_o, P_i).$
2.  $\mathbf{cd\_outcome} = \mathbf{LossAt}(\mathbf{c})$  and  $\varepsilon_D < D$  implies  $\mathbf{conflict?}(D, H, 0, T, P_o, P_i).$

**Theorem 7** (Only The Intruder Aircraft Is Turning). *For all  $P_o, P_i, P'_i, \mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \alpha_i, \omega_i, R_i, v_{iz}, \beta_i, n, \mathbf{cd\_outcome}, N, D, H$ , and  $T$ , if*

- $P_o = \mathbf{linear}_{\mathbf{s}_o, \mathbf{v}_o}$  (cf. Section 3.1),
- $P_i = \mathbf{turn}_{\mathbf{s}_i, \alpha_i, \omega_i, R_i, v_{iz}},$
- $\beta_i = \max(|\alpha_i|, |\alpha_i + T \cdot \omega_i|),$
- $n = \mathbf{exp\_term\_num}(\varepsilon_D / (2 R_i), \beta_i),$

- $P'_i = \text{turnpoly}_{s_i, \alpha_i, R_i, \omega_i, v_{iz}, n}$ , and
- $\text{cd\_outcome} = \text{cd\_poly}(D, H, P_o, P'_i, 0, T, N, \varepsilon_D, 0)$ ,

then

1.  $\text{cd\_outcome} = \text{NoConflict}$  implies  $\neg \text{conflict?}(D, H, 0, T, P_o, P_i)$ .
2.  $\text{cd\_outcome} = \text{LossAt}(c)$  and  $\varepsilon_D < D$  implies  $\text{conflict?}(D, H, 0, T, P_o, P_i)$ .

## 7 Conclusion

A polynomial global optimization method for developing verifiable conflict detection algorithms for arbitrary trajectories has been proposed. The key idea of the method is to compute bounds for the maximum and minimum values of a function that is defined as the maximum of two polynomials. The bounds are computed using a recursive branch-and-bound algorithm via Bernstein polynomials.

The proposed method has been specified and verified in the interactive theorem prover PVS, and it assumes that computations are performed using real number semantics with infinite precision, a built-in feature of PVS. In a programming language, this semantics can be achieved using a library for arbitrary precision arithmetic such as GMP.<sup>4</sup>

In contrast to other global optimization methods, the method proposed here has been mechanically verified in a theorem prover. Therefore, it is guaranteed to be free of logical errors. Further, if implemented in a programming language, the correctness of the software only depends on the correctness of the library for arbitrary precision arithmetic, which is relatively small compared to general purpose global optimization environments such as GlobSol<sup>5</sup> and COCONUT<sup>6</sup>. The use of a specialized algorithm, which has been formally verified, as opposed to a general tool for global optimization could greatly simplify the certification process needed for the deployment of safety-critical CD&R systems.

Branch-and-bound methods for global optimization are usually exponential in the number of variables, and the method presented here is not the exception. However, recent developments in branching and pruning heuristics and efficient representations of multivariate polynomials make Bernstein polynomial methods practical in many cases. Future research will study the feasibility of the proposed approach for airborne conflict detection, where the computational resources are scarce and the frequency of execution is on the order of 1Hz.

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<sup>4</sup><http://gmplib.org/>.

<sup>5</sup><http://interval.louisiana.edu/GlobSol>.

<sup>6</sup><http://www.mat.univie.ac.at/~coconut/coconut-environment>.

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